



***SPINTRONIC TECHNOLOGY  
&ADVANCE RESEARCH***

**LECTURE  
NOTES  
ON**

**HEAT TRANSFER**

**5<sup>TH</sup> SEMESTER B. Tech**

***BY  
ASST.PROF S.R.SAHOO***

# 5th Semester

## Heat Transfer

### MODULE-I (12 HOURS)

#### Introduction:

Modes of heat transfer: conduction, convection, and radiation, Mechanism & basic laws governing conduction, convection, and radiation heat transfer; Thermal conductivity, Thermal conductance & Thermal resistance, Contact resistance, convective heat transfer coefficient, radiation heat transfer coefficient, Electrical analogy, combined modes of heat transfer. Initial conditions and Boundary conditions of 1st, 2nd and 3rd Kind.

#### Heat Conduction:

The General heat conduction in Cartesian, polar-cylindrical and polar-spherical coordinates, Simplification of the general equation for one and two dimensional steady transient conduction with constant/ variable thermal conductivity with / without heat generation. Solution of the one dimensional steady state heat conduction problem in case of plane walls, cylinders and spheres for simple and composite cases. Critical insulation thickness, Heat transfer in extended surfaces (pin fins) without heat generation, Long fin, short fin with insulated tip and without insulated tip and fin connected between two heat sources. Fin efficiency and fin effectiveness. Conduction in solids with negligible internal temperature gradient (Lumped heat analysis).

### MODULE-II (12 HOURS)

#### Convective Heat Transfer:

Introduction to convective flow - forced and free. Dimensional analysis of forced and free convective heat transfer. Application of dimensional analysis, physical significance of Grashoff, Reynolds, Prandtl, Nusselt and Stanton numbers. Conservation equations for mass, momentum and energy for 2-dimensional convective heat transfer in case of incompressible flow, Hydrodynamic and thermal boundary layers for flow over a flat plate. Critical Reynolds number; general expressions for drag coefficient and drag force Reynolds- Colbourn analogy. Thermal boundary layer; general expression for local heat transfer coefficient; Average heat transfer Coefficient; Nusselt number. Flow inside a duct- velocity boundary layer, hydrodynamic entrance length and hydrodynamically developed flow; flow through tubes (internal flow). Use of empirical relations for solving turbulent conditions for external and internal flow. Mechanism of heat transfer during natural convection, Experimental heat transfer correlations for natural convection in the following cases (a) Vertical and horizontal plates (b) Inside and outside flows in case of tubes

### Module-III (8 HOURS)

#### Radiative heat exchange :

Introduction, Radiation properties, definitions of various terms used in radiation heat transfer; Absorptivity, reflectivity & transmissivity. Emissive power & emissivity, Kirchhoff's identity, Planck's relation for monochromatic emissive power of a black body, Derivation of Stefan-Boltzmann law and Wien's displacement law from Planck's relation, Radiation shape factor, Relation for shape factor and shape factor algebra. Heat exchange between blackbodies through non-absorbing medium. Gray bodies and real bodies, Heat exchange between gray bodies. Radiosity and Irradiation, Electrical analogy and radiation network for 2-body and 3-body radiations exchange in non-absorbing medium, Radiation shields.

### Module-IV (8 HOURS)

#### Heat transfer for boiling liquids and condensing vapours :

Types of condensation, use of correlations for condensation on vertical flat surfaces, horizontal tube and; regimes of pool boiling, pool boiling correlations. Critical heat flux, concept of forced boiling. Numerical problems.

#### Heat Exchangers :

Introduction, Types of heat exchanger, The overall heat transfer coefficient and fouling factors, LMTD and  $\epsilon$ -NTU analysis of heat exchangers.

#### Books :

- [1] Heat Transfer Incropera and Dewitt, Wiley publications
- [2] Heat Transfer : J.P. Holman, TMH Publications
- [3] Heat Transfer: P.S. Ghosdastidar, Oxford University Press
- [4] Fundamentals of Engineering Heat and Mass Transfer: R.C. Sachdeva, New Age International Publishers, 4th Edition

# **HEAT TRANSFER**

## **UNIT-I**

### **Introduction**

Heat is the form of energy that can be transferred from one system to another as a result of temperature difference. A thermodynamic analysis is concerned with the amount of heat transfer as a system undergoes a process from one equilibrium state to another. The science that deals with the determination of the rates of such energy transfers is the heat transfer. The transfer of energy as heat is always from the higher-temperature medium to the lower-temperature one, and heat transfer stops when the two mediums reach the same temperature.

Heat can be transferred in three different modes: conduction, convection, and radiation. All modes of heat transfer require the existence of a temperature difference and all modes are from the high-temperature medium to a lower-temperature one.

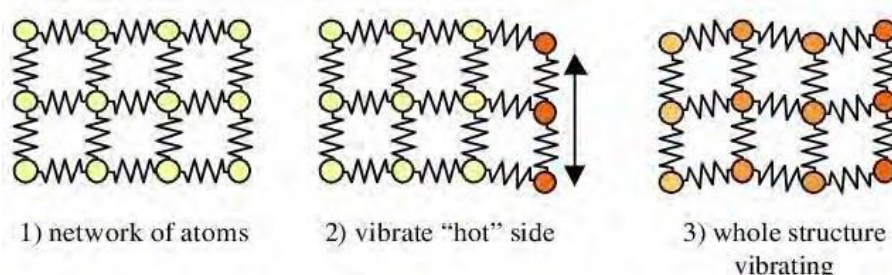
### **Three Modes of Heat Transfer**

There are three modes of heat transfer: conduction, convection, and radiation. Any energy exchange between bodies occurs through one of these modes or a combination of them. Conduction is the transfer of heat through solids or stationary fluids. Convection uses the movement of fluids to transfer heat. Radiation does not require a medium for transferring heat; this mode uses the electromagnetic radiation emitted by an object for exchanging heat.

### **Conduction**

Conduction is a transfer through solids or stationary fluids. When you touch a hot object, the heat you feel is transferred through your skin by conduction. Two mechanisms explain how heat is transferred by conduction: lattice vibration and particle collision. Conduction through solids occurs by a combination of the two mechanisms; heat is conducted through stationary fluids primarily by molecular collisions.

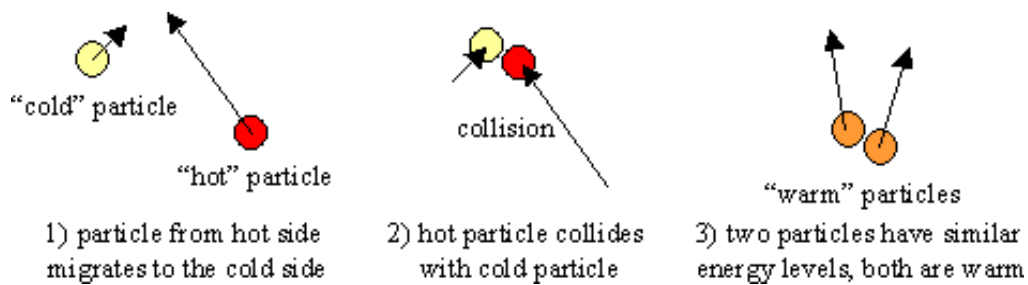
In solids, atoms are bound to each other by a series of bonds, analogous to springs. When there is a temperature difference in the solid, the hot side of the solid experiences more vigorous atomic movements. The vibrations are transmitted through the springs to the cooler side of the solid. Eventually they reach equilibrium, where all the atoms are vibrating with the same energy.



**Figure 1.1 Conduction by lattice vibration**

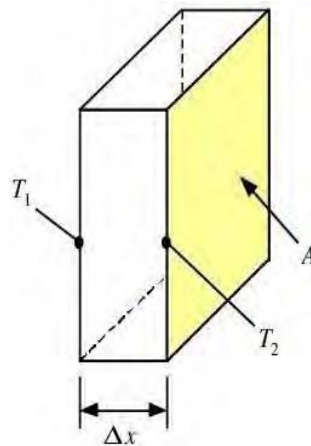
Solids, especially metals, have free electrons, which are not bound to any particular atom and can freely move about the solid. The electrons in the hot side of the solid move faster than those on the cooler side. This scenario is shown in Figure 1.2. As the electrons undergo a series of collisions, the faster electrons give off some of their energy to the slower electrons. Eventually, through a series of random collisions, equilibrium is reached, where the electrons are moving at the same average velocity. Conduction through electron collision is more effective than through lattice vibration; this is why metals generally are better heat conductors than ceramic materials, which do not have many free electrons.

In fluids, conduction occurs through collisions between freely moving molecules. The mechanism is identical to the electron collisions in metals. The effectiveness by which heat is transferred through a material is measured by the thermal conductivity,  $k$ . A good conductor, such as copper, has a high conductivity; a poor conductor, or an insulator, has a low conductivity.



**Figure 1.2 Conduction by particle collision**

Thermal Conductivity is measured in watts per meter per Kelvin (W/mK). The rate of heat transfer by conduction is given by:



**Figure 1.3 Heat Transfer by Conduction**

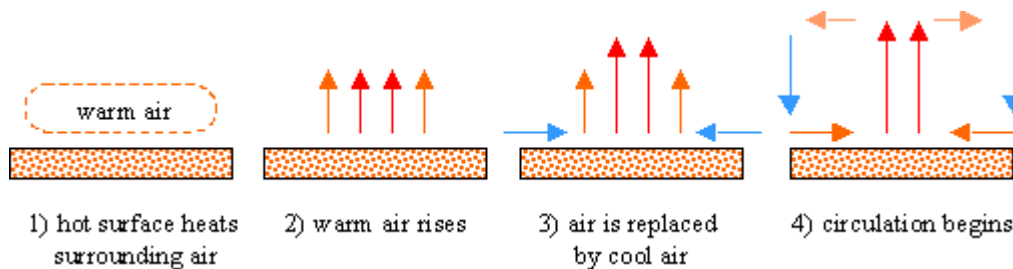
Where,  $A$  is the cross-sectional area, through which the heat is conducting,  $T$  is the temperature difference between the two surfaces separated by a distance  $\Delta x$  (see **Figure 1.3**). In heat transfer, a positive  $q$  means that heat is flowing into the body, and a negative  $q$  represents heat leaving the body. The negative sign in Eqn. 1.1

## Convection

Convection uses the motion of fluids to transfer heat. In a typical convective heat transfer, a hot surface heats the surrounding fluid, which is then carried away by fluid movement such as wind. The warm fluid is replaced by cooler fluid, which can draw more heat away from the surface. Since the heated fluid is constantly replaced by cooler fluid, the rate of heat transfer is enhanced.

*Natural convection* (or free convection) refers to a case where the fluid movement is created by the warm fluid itself. The density of fluid decrease as it is heated; thus, hot fluids are lighter than cool fluids. Warm fluid surrounding a hot object rises, and is replaced by cooler fluid. The result is a circulation of air above the warm surface, as shown in Figure 1.4.





**Figure 1.4 Natural convection**

Forced convection uses external means of producing fluid movement. Forced convection is what makes a windy, winter day feel much colder than a calm day with same temperature. The heat loss from your body is increased due to the constant replenishment of cold air by the wind. Natural wind and fans are the two most common sources of forced convection.

Convection coefficient,  $h$ , is the measure of how effectively a fluid transfers heat by convection. It is measured in  $W/m^2K$ , and is determined by factors such as the fluid density, viscosity, and velocity. Wind blowing at 5 mph has a lower  $h$  than wind at the same temperature blowing at 30 mph. The rate of heat transfer from a surface by convection is given by:

$$q_{convection} = -hA \cdot (T_{surface} - T_{\infty}) \quad (\text{Eq. 1.2})$$

Where,  $A$  is the surface area of the object,  $T_{surface}$  is the surface temperature, and  $T_{\infty}$  is the ambient or fluid temperature.

## Radiation

Radiative heat transfer does not require a medium to pass through; thus, it is the only form of heat transfer present in vacuum. It uses electromagnetic radiation (photons), which travels at the speed of light and is emitted by any matter with temperature above 0 degrees Kelvin ( $-273^{\circ}C$ ). Radiative heat transfer occurs when the emitted radiation strikes another body and is absorbed. We all experience radiative heat transfer everyday; solar radiation, absorbed by our skin, is why we feel warmer in the sun than in the shade.

The electromagnetic spectrum classifies radiation according to wavelengths of the radiation. Main types of radiation are (from short to long wavelengths): gamma rays, x-rays, ultraviolet (UV), visible light, infrared (IR), microwaves, and radio waves. Radiation with shorter wavelengths are more energetic and contains more heat. X-rays, having wavelengths  $\sim 10^{-9}$  m, are very energetic and can be harmful to humans, while visible light with wavelengths  $\sim 10^{-7}$  m contain less energy and therefore have little effect on life. A second characteristic which will become important later is that radiation with longer wavelengths generally can penetrate through thicker solids. Visible light, as we all know, is blocked by a wall. However, radio waves, having wavelengths on the order of meters, can readily pass through concrete walls.

Any body with temperature above 0 Kelvin emits radiation. The type of radiation emitted is determined largely by the temperature of the body. Most "hot" objects, from a cooking standpoint, emit infrared radiation. Hotter objects, such as the sun at  $\sim 5800$  K, emits more energetic radiation including visible and UV. The visible portion is evident from the bright glare of the sun; the UV radiation causes tans and burns. The amount of radiation emitted by an object is given by:

$$q_{emitted} = \epsilon \sigma \cdot AT^4 \quad (\text{Eq. 1.3})$$

Where,  $A$  is the surface area,  $T$  is the temperature of the body,  $\sigma$  is a constant called Stefan-Boltzmann constant, equal to  $5.67 \times 10^{-8} W/m^2K^4$ , and  $\epsilon$  is a material property called emissivity. The emissivity has a value between zero and 1, and is a measure of how efficiently a surface emits radiation. It is the ratio of the radiation emitted by a surface to the radiation emitted by a perfect emitter at the same temperature.

The emitted radiation strikes a second surface, where it is reflected, absorbed, or transmitted (Figure 1.5).

The portion that contributes to the heating of the surface is the absorbed radiation. The percentage of the incident radiation that is absorbed is called the absorptivity,  $\alpha$ .

The amount of heat absorbed by the surface is given by:

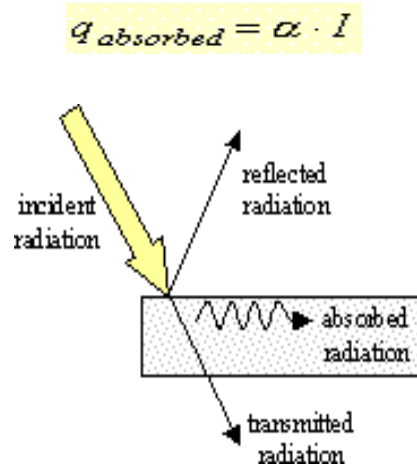


Figure 1.5 Interaction between a surface and incident radiation

Where,  $I$  is the incident radiation. The incident radiation is determined by the amount of radiation emitted by the object and how much of the emitted radiation actually strikes the surface. The latter is given by the shape factor,  $F$ , which is the percentage of the emitted radiation reaching the surface. The net amount of radiation absorbed by the surface is:

$$q_{\text{absorbed}} = F \cdot \alpha_2 \varepsilon_1 \sigma \cdot A_1 T^4 \quad (\text{Eq. 1.5})$$

For an object in an enclosure, the radiative exchange between the object and the wall is greatly simplified:

$$q_{\text{enclosure}} = -\sigma \varepsilon_{\text{object}} A_{\text{object}} (T_{\text{object}}^4 - T_{\text{wall}}^4) \quad (\text{Eq. 1.6})$$

This simplification can be made because all of the radiation emitted by the object strikes the wall ( $F_{\text{object} \rightarrow \text{wall}} = 1$ ).

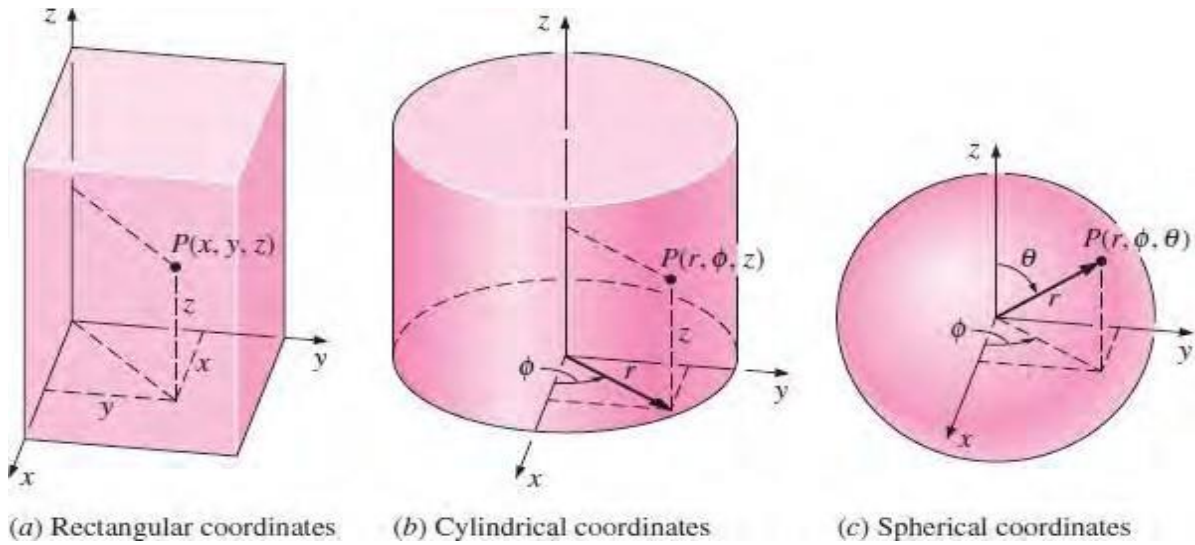
## Heat conduction Equation

Consider heat conduction through a large plane wall such as the wall of a house, the glass of a single pane window, the metal plate at the bottom of a pressing iron, a cast iron steam pipe, a cylindrical nuclear fuel element, an electrical resistance wire, the wall of a spherical container, or a spherical metal ball that is being quenched or tempered. Heat conduction in these and many other geometries can be approximated as being *one-dimensional* since heat conduction through these geometries will be dominant in one direction and negligible in other directions. Below we will develop the one-dimensional heat conduction equation in rectangular, cylindrical, and spherical coordinates.

Then,  $\left(\frac{\partial t}{\partial x}\right) dx$  = Change of temperature through distance  $dx$ , and

$t + \left(\frac{\partial t}{\partial x}\right) dx$  = temperature on the right face  $EFGH$  (at a distance  $dx$  from the left face  $ABCD$ ).

Further, let,  $k_x, k_y, k_z$  = Thermal conductivities (direction characteristics of the material) along  $X, Y$  and  $Z$  axes.



If the directional characteristics of a material are equal/same, it is called an “*Isotropic material*” and if unequal/different “*Anisotropic material*”.

$q_g$  = Heat generated per unit volume per unit time.

Inside the control volume there may be heat sources due to flow of electric current in electric motors and generators, nuclear fission etc.

(Note :  $q_g$  may be function of position or time, or both).

$\rho$  = Mass density of material, and

$c$  = Specific heat of the material.

#### **Energy balance/equation for volume element :**

Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered (A) + heat generated within the element (B) = Energy stored in the element (C). ... (1)

Let,  $Q$  = Rate of heat flow in a direction, and

$Q' = (Q \cdot d\tau)$  = Total heat flow (flux) in that direction (in time  $d\tau$ ).

A. Net heat accumulated in the element due to conduction of heat from all the directions considered:

Quantity of heat flowing into the element from the left face ABCD during the time interval  $d\tau$  in X-direction is given by :

$$\text{Heat influx,} \quad Q'_x = -k_x (dy \cdot dz) \frac{\partial t}{\partial x} \cdot d\tau$$

During the same time interval  $d\tau$  the heat flowing out of the right face of control volume ( $EFGH$ ) will be :

$$\text{Heat efflux, } Q'_{(x+dx)} = Q'_x + \frac{\partial}{\partial x} (Q'_x) dx \quad \dots(ii)$$

$\therefore$  Heat accumulation in the element due to heat flow in X-direction,

$$\begin{aligned} dQ'_x &= Q'_x - \left[ Q'_x + \frac{\partial}{\partial x} (Q'_x) dx \right] \quad [\text{Subtracting (ii) from (i)}] \\ &= - \frac{\partial}{\partial x} (Q'_x) dx \\ &= - \frac{\partial}{\partial x} \left[ -k_x (dy.dz) \frac{\partial t}{\partial x} . d\tau \right] dx \\ &= \frac{\partial}{\partial x} \left[ k_x \frac{\partial t}{\partial x} \right] dx.dy.dz.d\tau \quad \dots(2.1) \end{aligned}$$

Similarly the heat accumulated due to heat flow by conduction along Y and Z directions in time  $d\tau$  will be :

$$dQ'_y = \frac{\partial}{\partial y} \left[ k_y \frac{\partial t}{\partial y} \right] dx.dy.dz.d\tau \quad \dots(2.2)$$

$$dQ'_z = \frac{\partial}{\partial z} \left[ k_z \frac{\partial t}{\partial z} \right] dx.dy.dz.d\tau \quad \dots(2.3)$$

$\therefore$  Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered

$$\begin{aligned} &= \frac{\partial}{\partial x} \left[ k_x \frac{\partial t}{\partial x} \right] dx.dy.dz.d\tau + \frac{\partial}{\partial y} \left[ k_y \frac{\partial t}{\partial y} \right] dx.dy.dz.d\tau + \frac{\partial}{\partial z} \left[ k_z \frac{\partial t}{\partial z} \right] dx.dy.dz.d\tau \\ &= \left[ \frac{\partial}{\partial x} \left( k_x \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial t}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial t}{\partial z} \right) \right] dx.dy.dz.d\tau \quad \dots(2.4) \end{aligned}$$

B. Total heat generated within the element ( $Q'_g$ ) :

The total heat generated in the element is given by

$$Q'_g = q_g (dx.dy.dz) d\tau \quad \dots(2.5)$$

C. Energy stored in the element :

The total heat accumulated in the element due to heat flow along coordinate axes (Eqn. 2.4) and the heat generated within the element (Eqn. 2.5) together serve to increase the thermal energy of the element/lattice. This increase in thermal energy is given by

$$\rho (dx.dy.dz) c . \frac{\partial t}{\partial \tau} . d\tau \quad \dots(2.6)$$

[ $\therefore$  Heat stored in the body = Mass of the body  $\times$  specific heat of the body material  $\times$  rise in the temperature of body].

Now, substituting eqns. (2.4), (2.5), (2.6), in the eqn. (1), we have

$$\left[ \frac{\partial}{\partial x} \left( k_x \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial t}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial t}{\partial z} \right) \right] dx.dy.dz.d\tau + q_g (dx.dy.dz.)d\tau = \rho (dx.dy.dz) c . \frac{\partial t}{\partial \tau} . d\tau$$

Dividing both sides by  $dx.dy.dz.d\tau$ , we have

$$\frac{\partial}{\partial x} \left( k_x \frac{\partial t}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial t}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_z \frac{\partial t}{\partial z} \right) + q_g = \rho . c . \frac{\partial t}{\partial \tau} \quad \dots(2.7)$$

or, using the vector operator  $\nabla$ , we get

$$\nabla \cdot (k \nabla t) + q_g = \rho \cdot c \cdot \frac{\partial t}{\partial \tau} \quad \dots[2.7 (a)]$$

This is known as the **general heat conduction equation for ‘non-homogeneous material’, ‘self heat generating’ and ‘unsteady three-dimensional heat flow’**. This equation establishes in *differential form the relationship between the time and space variation of temperature at any point of solid through which heat flow by conduction takes place.*

**General heat conduction equation for constant thermal conductivity :**

In case of homogeneous (in which properties *e.g.*, specific heat, density, thermal conductivity etc. are same everywhere in the material) and isotropic (in which properties are independent of surface orientation) material,  $k_x = k_y = k_z = k$  and diffusion equation Eqn. (2.7) becomes

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{\rho \cdot c}{k} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots(2.8)$$

where,

$$\alpha = \frac{k}{\rho \cdot c} = \frac{\text{Thermal conductivity}}{\text{Thermal capacity}}$$

The quantity,  $\alpha = \frac{k}{\rho \cdot c}$  is known as **thermal diffusivity**.

- The larger the value of  $\alpha$ , the faster will the heat diffuse through the material and its temperature will change with time. This will result either due to a high value of thermal conductivity  $k$  or a low value of heat capacity  $\rho \cdot c$ . A low value of heat capacity means

the less amount of heat entering the element, would be absorbed and used to raise its temperature and more would be available for onward transmission. Metals and gases have relatively high value of  $\alpha$  and their response to temperature changes is quite rapid. The non-metallic solids and liquids respond slowly to temperature changes because of their relatively small value of thermal diffusivity.

- Thermal diffusivity is an important characteristic quantity for *unsteady conduction situations*.

Eqn. (2.8) by using Laplacian  $\nabla^2$ , may be written as :

$$\nabla^2 t + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots[2.8 (a)]$$

Eqn. (2.8), governs the temperature distribution under unsteady heat flow through a material which is homogeneous and isotropic.

**Other simplified forms of heat conduction equation in cartesian coordinates :**

- (i) For the case when *no internal source of heat generation is present*, Eqn. (2.8) reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \left[ \text{Unsteady state } \left( \frac{\partial t}{\partial \tau} \neq 0 \right) \text{ heat flow with no internal heat generation} \right]$$

or,

$$\nabla^2 t = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \textbf{(Fourier's equation)} \quad \dots(2.9)$$

- (ii) Under the situations when temperature does not depend on time, the conduction then takes place in the steady state  $\left( i.e., \frac{\partial t}{\partial \tau} = 0 \right)$  and the eqn. (2.8) reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = 0$$

or,

$$\nabla^2 t + \frac{q_g}{k} = 0 \quad \textbf{(Poisson's equation)} \quad \dots(2.10)$$

In the absence of internal heat generation, Eqn. (2.10) reduces to

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0$$

or,

$$\nabla^2 t = 0 \quad \textbf{(Laplace equation)} \quad \dots(2.11)$$



(iii) *Steady state and one-dimensional heat transfer:*

$$\frac{\partial^2 t}{\partial x^2} + \frac{q_g}{k} = 0 \quad \dots(2.12)$$

(iv) *Steady state, one-dimensional, without internal heat generation*

$$\frac{\partial^2 t}{\partial x^2} = 0 \quad \dots(2.13)$$

(v) *Steady state, two dimensional, without internal heat generation*

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \quad \dots(2.14)$$

(vi) *Unsteady state, one dimensional, without internal heat generation*

$$\frac{\partial^2 t}{\partial x^2} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots(2.15)$$

While dealing with problems of conduction of heat through systems having cylindrical geometries (e.g., rods and pipes) it is convenient to use cylindrical coordinates.

Consider an elemental volume having the coordinates ( $r, \phi, z$ ), for three-dimensional heat conduction analysis, as shown in Fig. 2.2.

The volume of the element =  $rd\phi.dr.dz$

Let,  $q_g$  = Heat generation (uniform) per unit volume per unit time.

Further, let us assume that  $k$  (thermal conductivity),  $\rho$  (density),  $c$  (specific heat) do not alter with position.

A. *Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered :*

Heat flow in radial direction ( $x-\phi$ ) plane :

$$\text{Heat influx,} \quad Q'_r = -k (rd\phi.dz) \frac{\partial t}{\partial r} \cdot d\tau \quad \dots(i)$$

$$\text{Heat efflux,} \quad Q'_{(r+dr)} = Q'_r + \frac{\partial}{\partial r} (Q'_r) dr \quad \dots(ii)$$

$\therefore$  Heat accumulation in the element due to heat flow in *radial direction*,

$$\begin{aligned} dQ'_r &= Q'_r - Q'_{(r+dr)} && [\text{subtracting (ii) from (i)}] \\ &= - \frac{\partial}{\partial r} (Q'_r) dr \\ &= - \frac{\partial}{\partial r} \left[ -k (rd\phi.dz) \frac{\partial t}{\partial r} \cdot d\tau \right] dr \\ &= k (dr.d\phi.dz) \frac{\partial}{\partial r} \left( r \cdot \frac{\partial t}{\partial r} \right) d\tau \\ &= k (dr.d\phi.dz) \left( r \frac{\partial^2 t}{\partial r^2} + \frac{\partial t}{\partial r} \right) d\tau \end{aligned}$$

$$= k (dr.rd\phi.dz) \left[ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} \right] d\tau \quad \dots(2.16)$$

Heat flow in *tangential direction* ( $r$ - $z$ ) plane :

$$\text{Heat influx,} \quad Q'_\phi = -k (dr.dz) \frac{\partial t}{r.\partial\phi} d\tau \quad \dots(iii)$$

$$\text{Heat efflux,} \quad Q'_{(\phi+d\phi)} = Q'_\phi + \frac{\partial}{r.\partial\phi} (Q'_\phi) r d\phi \quad \dots(iv)$$

Heat accumulated in the element due to heat flow in *tangential direction*,

$$dQ'_\phi = Q'_\phi - Q'_{(\phi+d\phi)} \quad [\text{subtracting (iv) from (iii)}]$$

$$\begin{aligned} dQ'_\phi &= Q'_\phi - Q'_{(\phi+d\phi)} \\ &= - \frac{\partial}{r.\partial\phi} (Q'_\phi) r.d\phi \\ &= - \frac{\partial}{r.\partial\phi} \left[ -k (dr.dz) \frac{\partial t}{r.\partial\phi} . d\tau \right] r.d\phi \\ &= k (dr.d\phi.dz) \frac{\partial}{\partial\phi} \left( \frac{1}{r} . \frac{\partial t}{\partial\phi} \right) d\tau \\ &= k (dr.rd\phi.dz) \frac{1}{r^2} . \frac{\partial^2 t}{\partial\phi^2} . d\tau \end{aligned}$$

Heat flow in *axial direction* ( $r$ - $\phi$  plane) :

$$\text{Heat influx,} \quad Q'_z = -k (r.d\phi.dr) \frac{\partial t}{\partial z} d\tau \quad \dots(v)$$

$$\text{Heat efflux,} \quad Q'_{(z+dz)} = Q'_z + \frac{\partial}{\partial z} (Q'_z) dz \quad \dots(vi)$$

Heat accumulated in the element due to heat flow in *axial direction*,

$$dQ'_z = Q'_z - Q'_{(z+dz)}$$



$$\begin{aligned}
&= -\frac{\partial}{\partial z} \left[ -k (r.d\phi.dr) \frac{\partial t}{\partial z} . d\tau \right] dz \\
&= k (dr.rd\phi.dz) \frac{\partial^2 t}{\partial z^2} . d\tau \quad \dots(2.18)
\end{aligned}$$

Net heat accumulated in the element

$$= k.dr.rd\phi.dz \left[ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \right] d\tau \quad \dots(2.19)$$

B. Heat generated within the element ( $Q'_g$ ) :

The total heat generated within the element is given by

$$Q'_g = q_g (dr.rd\phi.dz).d\tau \quad \dots(2.20)$$

C. Energy stored in the element :

The increase in thermal energy in the element is equal to

$$= \rho(dr.rd\phi.dz).c.\frac{\partial t}{\partial \tau} . d\tau \quad \dots(2.21)$$

Now, (A) + (B) = (C) ... Energy balance/equation

$$\begin{aligned}
\therefore k.dr.rd\phi.dz \left[ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \right] d\tau + q_g (dr.rd\phi.dz).d\tau \\
= \rho(dr.rd\phi.dz).c.\frac{\partial t}{\partial \tau} . d\tau
\end{aligned}$$

Dividing both sides by  $dr.rd\phi.dz.d\tau$ , we have

$$\begin{aligned}
&k \left[ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \right] + q_g = \rho.c. \frac{\partial t}{\partial \tau} \\
\text{or, } &\left[ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \right] + \frac{q_g}{k} = \frac{\rho.c}{k} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots(2.22)
\end{aligned}$$

Equation (2.22) is the **general heat conduction equation in cylindrical coordinates**.

In case there are no *heat sources present* and the heat flow is *steady* and *one-dimensional*, then eqn. (2.22) reduces to

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} = 0 \quad \dots(2.23)$$

$$\text{or, } \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{dt}{dr} = 0$$

$$\text{or, } \frac{1}{r} \cdot \frac{d}{dr} \left( r \cdot \frac{dt}{dr} \right) = 0$$

Since  $\frac{1}{r} \neq 0$ , therefore,

$$\frac{d}{dr} \left( r \cdot \frac{dt}{dr} \right) \text{ or } r \cdot \frac{dt}{dr} = \text{constant} \quad \dots(2.24)$$

Equation (2.22) can also be derived by transformation of coordinates, as follows :

$$x = r \cos \phi, y = r \sin \phi \text{ and } z = z$$

Now, by chain rule :

$$\frac{\partial t}{\partial r} = \frac{\partial t}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial t}{\partial x} \cos \phi + \frac{\partial t}{\partial y} \sin \phi$$

or,  $\cos \phi \frac{\partial t}{\partial r} = \cos^2 \phi \cdot \frac{\partial t}{\partial x} + \sin \phi \cdot \cos \phi \cdot \frac{\partial t}{\partial y}$  ... (i)

(Multiplying both sides by  $\cos \phi$ )

Also,  $\frac{\partial t}{\partial \phi} = \frac{\partial t}{\partial x} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial \phi} = \frac{\partial t}{\partial x} (-r \sin \phi) + \frac{\partial t}{\partial y} (r \cos \phi)$

or,  $\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} = -\sin^2 \phi \frac{\partial t}{\partial x} + \sin \phi \cdot \cos \phi \cdot \frac{\partial t}{\partial y}$  ... (ii)

(Multiplying both sides by  $\frac{\sin \phi}{r}$ )

From Eqns. (i) and (ii), we have

$$\begin{aligned} \frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} &= -\sin^2 \phi \frac{\partial t}{\partial x} + \left[ \cos \phi \cdot \frac{\partial t}{\partial r} - \cos^2 \phi \frac{\partial t}{\partial x} \right] \\ &= -\frac{\partial t}{\partial x} + \cos \phi \frac{\partial t}{\partial r} \end{aligned}$$

$\therefore \frac{\partial t}{\partial x} = \cos \phi \frac{\partial t}{\partial r} - \frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}$  ... (iii)

Differentiating both sides with respect to  $x$ , we have

$$\frac{\partial}{\partial x} \left( \frac{\partial t}{\partial x} \right) = \frac{\partial}{\partial x} \left[ \cos \phi \cdot \frac{\partial t}{\partial r} - \frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} \right]$$

or,  $\frac{\partial^2 t}{\partial x^2} = \cos \phi \cdot \frac{\partial}{\partial r} \left( \frac{\partial t}{\partial x} \right) - \frac{\sin \phi}{r} \cdot \frac{\partial}{\partial \phi} \left( \frac{\partial t}{\partial x} \right)$

$$= \cos \phi \cdot \frac{\partial}{\partial r} \left( \cos \phi \cdot \frac{\partial t}{\partial r} - \frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} \right) - \frac{\sin \phi}{r} \cdot \frac{\partial}{\partial \phi} \left( \cos \phi \cdot \frac{\partial t}{\partial r} - \frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} \right)$$

[Substituting the value of  $\frac{\partial t}{\partial x}$  from (iii)]

$$= \cos^2 \phi \cdot \frac{\partial^2 t}{\partial r^2} - \frac{\cos \phi \cdot \sin \phi}{r^2} \cdot \frac{\partial t}{\partial \phi} + \frac{\sin^2 \phi}{r} \cdot \frac{\partial t}{\partial r} + \frac{\sin^2 \phi}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\sin \phi \cdot \cos \phi}{r^2} \cdot \frac{\partial t}{\partial \phi}$$

... (iv)

Similarly,  $\frac{\partial^2 t}{\partial y^2} = \sin^2 \phi \cdot \frac{\partial^2 t}{\partial r^2} + \frac{\cos^2 \phi}{r} \cdot \frac{\partial t}{\partial r} - \frac{\cos \phi \cdot \sin \phi}{r^2} \cdot \frac{\partial t}{\partial \phi} + \frac{\cos^2 \phi}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} - \frac{\cos \phi \cdot \sin \phi}{r^2} \cdot \frac{\partial t}{\partial \phi}$

... (v)

By adding (iii) and (iv), we get

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2}$$

Substituting it in eqn (2.8), we get,

$$\left[ \frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2} \right] + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}$$

which is the same as eqn. (2.22)

Consider an elemental volume having the coordinates  $(r, \phi, \theta)$ , for three dimensional heat conduction analysis, as shown in Fig. 2.3.

The volume of the element =  $dr \cdot r d\theta \cdot r \sin \theta d\phi$

Let,  $q_g$  = Heat generation (uniform) per unit volume per unit time.

Further let us assume that  $k$  (thermal conductivity),  $\rho$  (density),  $c$  (specific heat) do not alter with position.

A. Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered :

Heat flow through  $r$ - $\theta$  plane;  $\phi$ -direction :

Heat influx,  $Q'_\phi = -k (dr \cdot r d\theta) \frac{\partial t}{r \cdot \sin \theta \cdot \partial \phi} d\tau \quad \dots(i)$

Heat efflux,  $Q'_{(\phi + d\phi)} = Q'_\phi + \frac{\partial}{\partial \phi} (Q'_\phi) r \sin \theta \cdot d\phi \quad \dots(ii)$

$\therefore$  Heat accumulated in the element due to heat flow in the  $\phi$ -direction,

$$\begin{aligned} dQ'_\phi &= Q'_\phi - Q'_{(\phi + d\phi)} \quad [\text{subtracting (ii) from (i)}] \\ &= -\frac{1}{r \sin \theta} \cdot \frac{\partial}{\partial \phi} (Q'_\phi) r \sin \theta \cdot d\phi \\ &= -\frac{1}{r \sin \theta} \cdot \frac{\partial}{\partial \phi} \left[ -k (dr \cdot r d\theta) \frac{1}{r \sin \theta} \cdot \frac{\partial t}{\partial \phi} \cdot d\tau \right] r \sin \theta \cdot d\phi \\ &= k (dr \cdot r d\theta \cdot r \sin \theta \cdot d\phi) \frac{1}{r^2 \sin^2 \theta} \cdot \frac{\partial^2 t}{\partial \phi^2} d\tau \quad \dots(2.25) \end{aligned}$$

Heat flow in  $r$ - $\phi$  plane,  $\theta$ -direction :

Heat influx,  $Q'_\theta = -k (dr \cdot r \sin \theta \cdot d\phi) \frac{\partial t}{r d\theta} \cdot d\tau \quad \dots(iii)$

Heat efflux,  $Q'_{(\theta + d\theta)} = Q'_\theta + \frac{\partial}{\partial \theta} (Q'_\theta) r d\theta \quad \dots(iv)$

$\therefore$  Heat accumulated in the element due to heat flow in the  $\theta$ -direction,

$$\begin{aligned} dQ'_\theta &= Q'_\theta - Q'_{(\theta + d\theta)} \quad [\text{subtracting (iv) from (iii)}] \\ &= -\frac{\partial}{\partial \theta} (Q'_\theta) r \cdot d\theta \\ &= -\frac{\partial}{\partial \theta} \left[ -k (dr \cdot r \sin \theta \cdot d\phi) \frac{\partial t}{r \cdot \partial \theta} \cdot d\tau \right] r \cdot d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{k}{r} \frac{dr \cdot r d\phi \cdot r d\theta}{r} \frac{\partial}{\partial \theta} \left[ \sin \theta \cdot \frac{\partial t}{\partial \theta} \right] d\tau \\
&= k (dr \cdot r d\theta \cdot r \sin \theta \cdot d\phi) \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left[ \sin \theta \cdot \frac{\partial t}{\partial \theta} \right] d\tau \quad \dots(2.26)
\end{aligned}$$

Heat flow in  $\theta$ - $\phi$  plane,  $r$ -direction :

$$\text{Heat influx, } Q'_r = -k (rd\theta \cdot r \sin \theta \cdot d\phi) \frac{\partial t}{\partial r} \cdot d\tau \quad \dots(v)$$

$$\text{Heat efflux, } Q'_{(r+dr)} = Q'_r + \frac{\partial}{\partial r} (Q'_r) dr \quad \dots(vi)$$

$\therefore$  Heat accumulation in the element due to heat flow in the  $r$ -direction,

$$\begin{aligned}
dQ'_r &= Q'_r - Q'_{(r+dr)} \quad [\text{subtracting (vi) from (v)}] \\
&= -\frac{\partial}{\partial r} (Q'_r) dr \\
&= -\frac{\partial}{\partial r} \left[ -k (rd\theta \cdot r \sin \theta \cdot d\phi) \frac{\partial t}{\partial r} \cdot d\tau \right] dr \\
&= k d\theta \cdot \sin \theta \cdot d\phi dr \frac{\partial}{\partial r} \left[ r^2 \cdot \frac{\partial t}{\partial r} \right] d\tau \\
&= k (dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi) \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left[ r^2 \cdot \frac{\partial t}{\partial r} \right] d\tau \quad \dots(2.27)
\end{aligned}$$

Net heat accumulated in the element

$$= k dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi \left[ \frac{1}{r^2 \sin^2 \theta} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left( \sin \theta \cdot \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left( r^2 \cdot \frac{\partial t}{\partial r} \right) \right] d\tau \quad \dots(2.28)$$

B. Heat generated within the element ( $Q'_g$ ) :

The total heat generated within the element is given by,

$$Q'_g = q_g (dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi) d\tau \quad \dots(2.29)$$

C. Energy stored in the element :

The increase in thermal energy in the element is equal to

$$\rho (dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi) c \cdot \frac{\partial t}{\partial \tau} \cdot d\tau \quad \dots(2.30)$$

Now, (A) + (B) = (C) ...Energy balance/equation

$$\begin{aligned}
\therefore k dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi \left[ \frac{1}{r^2 \sin^2 \theta} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left( \sin \theta \cdot \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left( r^2 \cdot \frac{\partial t}{\partial r} \right) \right] \cdot d\tau \\
+ q_g (dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi) d\tau = \rho (dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi) c \cdot \frac{\partial t}{\partial \tau} \cdot d\tau
\end{aligned}$$

Dividing both sides by  $k(dr \cdot rd\theta \cdot r \sin \theta \cdot d\phi)d\tau$ , we get

$$\begin{aligned}
&\left[ \frac{1}{r^2 \sin^2 \theta} \cdot \frac{\partial^2 t}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left( \sin \theta \cdot \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left( r^2 \cdot \frac{\partial t}{\partial r} \right) \right] + \frac{q_g}{k} \\
&= \frac{\rho c}{k} \cdot \frac{\partial t}{\partial \tau} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots(2.31)
\end{aligned}$$

Equation (2.31) is the **general heat conduction equation in spherical coordinates**.

In case there are not *heat sources present* and the heat flow is *steady* and *one-dimensional*, then eqn. (2.31) reduces to

$$\frac{1}{r^2} \cdot \frac{d}{dr} \left( r^2 \cdot \frac{dt}{dr} \right) = 0 \quad \dots(2.32)$$

Equation (2.31) can also be derived by transformation of coordinates as follows :

$$x = r \sin \theta \sin \phi ; y = r \sin \theta \cos \phi ; z = r \cos \theta$$

### Case I : Uniform thermal conductivity

Refer to Fig. 2.4 (a) Consider a plane wall of homogeneous material through which heat is flowing *only in x-direction*.

Let,  $L$  = Thickness of the plane wall,  
 $A$  = Cross-sectional area of the wall,  
 $k$  = Thermal conductivity of the wall material, and  
 $t_1, t_2$  = Temperatures maintained at the two faces 1 and 2 of the wall, respectively.

The general heat conduction equation in cartesian coordinates is given by

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_g}{k} = \frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \quad \dots[\text{Eqn. 2.8}]$$

If the heat conduction takes place under the

$\left[ \frac{\partial^2 t}{\partial y^2} = \frac{\partial^2 t}{\partial z^2} = 0 \right]$  and with no internal heat generation  $\left( \frac{q_g}{k} = 0 \right)$  then the above equation is reduced to

$$\frac{\partial^2 t}{\partial x^2} = 0, \quad \text{or} \quad \frac{d^2 t}{dx^2} = 0 \quad \dots(2.33)$$

By integrating the above differential twice, we have

$$\frac{dt}{dx} = C_1 \quad \text{and} \quad t = C_1 x + C_2 \quad \dots(2.34)$$

where  $C_1$  and  $C_2$  are the arbitrary constants. The values of these constants may be calculated from the known boundary conditions as follows :

$$\begin{array}{ll} \text{At } x = 0 & t = t_1 \\ \text{At } x = L & t = t_2 \end{array}$$

Substituting the values in the eqn. (2.34), we get

$$t_1 = 0 + C_2 \quad \text{and} \quad t_2 = C_1 L + C_2$$

After simplification, we have,  $C_2 = t_1$  and  $C_1 = \frac{t_2 - t_1}{L}$

Thus, the eqn. (2.34) reduces to :

$$t = \left( \frac{t_2 - t_1}{L} \right) x + t_1 \quad \dots(2.35)$$

The eqn. (2.35) indicates that *temperature distribution across a wall is linear* and is *independent of thermal conductivity*. Now heat through the plane wall can be found by using Fourier's equation as follows :

$$Q = -kA \frac{dt}{dx} \quad \left(\text{where, } \frac{dt}{dx} = \text{Temperature gradient}\right) \quad \dots$$

[Eqn.(1.1)]

But,

$$\frac{dt}{dx} = \frac{d}{dx} \left[ \left( \frac{t_2 - t_1}{L} \right) x + t_1 \right] = \frac{t_2 - t_1}{L}$$

$$\therefore Q = -kA \frac{(t_2 - t_1)}{L} = \frac{kA (t_1 - t_2)}{L} \quad \dots(2.36)$$

Eqn (2.36) can be written as :

$$Q = \frac{(t_1 - t_2)}{(L/kA)} = \frac{(t_1 - t_2)}{(R_{th})_{cond.}} \quad \dots(2.37)$$

where,  $(R_{th})_{cond.}$  = Thermal resistance to heat conduction. Fig. 2.4 (b) shows the *equivalent thermal circuit* for heat flow through the plane wall.

Let us now find out the condition when instead of space, weight is the main criterion for selection of the insulation of a plane wall.

Thermal resistance (conduction) of the wall,  $(R_{th})_{cond.} = \frac{L}{kA} \quad \dots(i)$

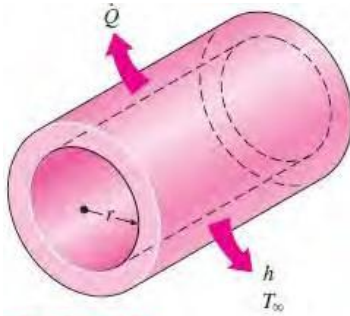
Weight of the wall,  $W = \rho A L \quad \dots(ii)$

Eliminating  $L$  from (i) and (ii), we get

$$W = \rho A. (R_{th})_{cond.} kA = (\rho.k)A^2.(R_{th})_{cond.} \quad \dots(2.38)$$

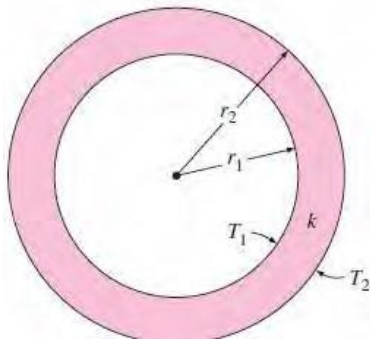
The eqn., (2.38) stipulates the condition that, for a specified thermal resistance, the *lightest insulation will be one which has the smallest product of density ( $\rho$ ) and thermal conductivity ( $k$ )*.

### Heat Conduction through Composite Wall:



**FIGURE 3-23**

Heat is lost from a hot water pipe to the air outside in the radial direction, and thus heat transfer from a long pipe is one-dimensional.



**FIGURE 3-24**

A long cylindrical pipe (or spherical shell) with specified inner and outer surface temperatures  $T_1$  and  $T_2$ .

### 3-4 ■ HEAT CONDUCTION IN CYLINDERS AND SPHERES

Consider steady heat conduction through a hot water pipe. Heat is continuously lost to the outdoors through the wall of the pipe, and we intuitively feel that heat transfer through the pipe is in the normal direction to the pipe surface and no significant heat transfer takes place in the pipe in other directions (Fig. 3-23). The wall of the pipe, whose thickness is rather small, separates two fluids at different temperatures, and thus the temperature gradient in the radial direction will be relatively large. Further, if the fluid temperatures inside and outside the pipe remain constant, then heat transfer through the pipe is *steady*. Thus heat transfer through the pipe can be modeled as *steady* and *one-dimensional*. The temperature of the pipe in this case will depend on one direction only (the radial  $r$ -direction) and can be expressed as  $T = T(r)$ . The temperature is independent of the azimuthal angle or the axial distance. This situation is approximated in practice in long cylindrical pipes and spherical containers.

In *steady* operation, there is no change in the temperature of the pipe with time at any point. Therefore, the rate of heat transfer into the pipe must be equal to the rate of heat transfer out of it. In other words, heat transfer through the pipe must be constant,  $\dot{Q}_{\text{cond, cyl}} = \text{constant}$ .

Consider a long cylindrical layer (such as a circular pipe) of inner radius  $r_1$ , outer radius  $r_2$ , length  $L$ , and average thermal conductivity  $k$  (Fig. 3-24). The two surfaces of the cylindrical layer are maintained at constant temperatures  $T_1$  and  $T_2$ . There is no heat generation in the layer and the thermal conductivity is constant. For one-dimensional heat conduction through the cylindrical layer, we have  $T(r)$ . Then Fourier's law of heat conduction for heat transfer

through the cylindrical layer can be expressed as

$$\dot{Q}_{\text{cond, cyl}} = -kA \frac{dT}{dr} \quad (\text{W}) \quad (3-35)$$

where  $A = 2\pi rL$  is the heat transfer area at location  $r$ . Note that  $A$  depends on  $r$ , and thus it *varies* in the direction of heat transfer. Separating the variables in the above equation and integrating from  $r = r_1$ , where  $T(r_1) = T_1$ , to  $r = r_2$ , where  $T(r_2) = T_2$ , gives

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A} dr = - \int_{T=T_1}^{T_2} k dT \quad (3-36)$$

Substituting  $A = 2\pi rL$  and performing the integrations give

$$\dot{Q}_{\text{cond, cyl}} = 2\pi Lk \frac{T_1 - T_2}{\ln(r_2/r_1)} \quad (\text{W}) \quad (3-37)$$

since  $\dot{Q}_{\text{cond, cyl}} = \text{constant}$ . This equation can be rearranged as

$$\dot{Q}_{\text{cond, cyl}} = \frac{T_1 - T_2}{R_{\text{cyl}}} \quad (\text{W}) \quad (3-38)$$



where

$$R_{\text{cyl}} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius/Inner radius})}{2\pi \times (\text{Length}) \times (\text{Thermal conductivity})} \quad (3-39)$$

is the *thermal resistance* of the cylindrical layer against heat conduction, or simply the **conduction resistance** of the cylinder layer.

We can repeat the analysis above for a *spherical layer* by taking  $A = 4\pi r^2$  and performing the integrations in Eq. 3–36. The result can be expressed as

$$\dot{Q}_{\text{cond. sph}} = \frac{T_1 - T_2}{R_{\text{sph}}} \quad (3-40)$$

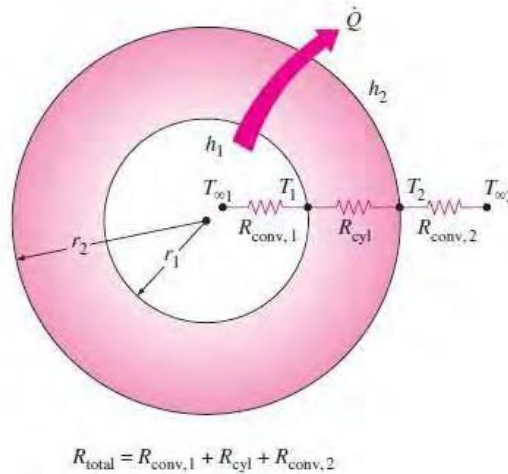
where

$$R_{\text{sph}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{Outer radius} - \text{Inner radius}}{4\pi(\text{Outer radius})(\text{Inner radius})(\text{Thermal conductivity})} \quad (3-41)$$

is the *thermal resistance* of the spherical layer against heat conduction, or simply the **conduction resistance** of the spherical layer.

Now consider steady one-dimensional heat flow through a cylindrical or spherical layer that is exposed to convection on both sides to fluids at temperatures  $T_{\infty 1}$  and  $T_{\infty 2}$  with heat transfer coefficients  $h_1$  and  $h_2$ , respectively, as shown in Fig. 3–25. The thermal resistance network in this case consists of one conduction and two convection resistances in series, just like the one for the plane wall, and the rate of heat transfer under steady conditions can be expressed as

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} \quad (3-42)$$



**FIGURE 3–25**

The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

where

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl}} + R_{\text{conv},2} \\ = \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi Lk} + \frac{1}{(2\pi r_2 L)h_2} \quad (3-43)$$

for a *cylindrical* layer, and

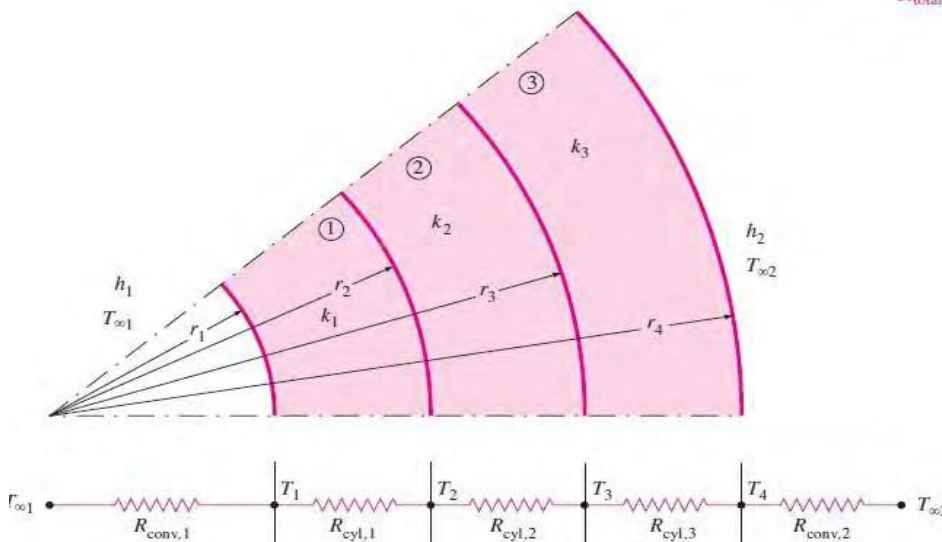
$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{sph}} + R_{\text{conv},2} \\ = \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2} \quad (3-44)$$

for a *spherical* layer. Note that  $A$  in the convection resistance relation  $R_{\text{conv}} = 1/hA$  is the *surface area at which convection occurs*. It is equal to  $A = 2\pi rL$  for a cylindrical surface and  $A = 4\pi r^2$  for a spherical surface of radius  $r$ . Also note that the thermal resistances are in series, and thus the total thermal resistance is determined by simply adding the individual resistances, just like the electrical resistances connected in series.

## Multilayered Cylinders and Spheres

Steady heat transfer through multilayered cylindrical or spherical shells can be handled just like multilayered plane walls discussed earlier by simply adding an *additional resistance* in series for each *additional layer*. For example, the steady heat transfer rate through the three-layered composite cylinder of length  $L$  shown in Fig. 3–26 with convection on both sides can be expressed as

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}} \quad (3-45)$$



**FIGURE 3–26**

The thermal resistance network for heat transfer through a three-layered composite cylinder subjected to convection on both sides.

where  $R_{\text{total}}$  is the *total thermal resistance*, expressed as

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl},1} + R_{\text{cyl},2} + R_{\text{cyl},3} + R_{\text{conv},2}$$

$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4} \quad (3-46)$$

where  $A_1 = 2\pi r_1 L$  and  $A_4 = 2\pi r_4 L$ . Equation 3-46 can also be used for a three-layered spherical shell by replacing the thermal resistances of cylindrical layers by the corresponding spherical ones. Again, note from the thermal resistance network that the resistances are in series, and thus the total thermal resistance is simply the *arithmetic sum* of the individual thermal resistances in the path of heat flow.

Once  $\dot{Q}$  is known, we can determine any intermediate temperature  $T_j$  by applying the relation  $\dot{Q} = (T_i - T_j)/R_{\text{total}, i-j}$  across any layer or layers such that  $T_i$  is a *known* temperature at location  $i$  and  $R_{\text{total}, i-j}$  is the total thermal resistance between locations  $i$  and  $j$  (Fig. 3-27). For example, once  $\dot{Q}$  has been calculated, the interface temperature  $T_2$  between the first and second cylindrical layers can be determined from

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv},1} + R_{\text{cyl},1}} = \frac{T_{\infty 1} - T_2}{\frac{1}{h_1(2\pi r_1 L)} + \frac{\ln(r_2/r_1)}{2\pi L k_1}} \quad (3-47)$$

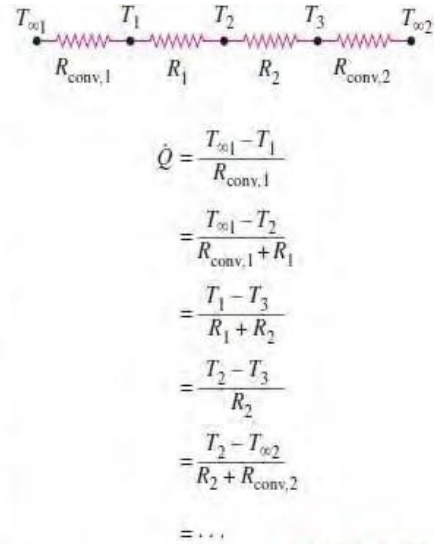
We could also calculate  $T_2$  from

We could also calculate  $T_2$  from

$$\dot{Q} = \frac{T_2 - T_{\infty 2}}{R_2 + R_3 + R_{\text{conv},2}} = \frac{T_2 - T_{\infty 2}}{\frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_o(2\pi r_4 L)}} \quad (3-48)$$

Although both relations will give the same result, we prefer the first one since it involves fewer terms and thus less work.

The thermal resistance concept can also be used for *other geometries*, provided that the proper conduction resistances and the proper surface areas in convection resistances are used.



**FIGURE 3-27**

The ratio  $\Delta T/R$  across any layer is equal to  $\dot{Q}$ , which remains constant in one-dimensional steady conduction.



## Critical Radius of Insulation

We know that adding more insulation to a wall or to the attic always decreases heat transfer. The thicker the insulation, the lower the heat transfer rate. This is expected, since the heat transfer area  $A$  is constant, and adding insulation always increases the thermal resistance of the wall without increasing the convection resistance.

Adding insulation to a cylindrical pipe or a spherical shell, however, is a different matter. The additional insulation increases the conduction resistance of the insulation layer but decreases the convection resistance of the surface because of the increase in the outer surface area for convection. The heat transfer from the pipe may increase or decrease, depending on which effect dominates.

Consider a cylindrical pipe of outer radius  $r_1$  whose outer surface temperature  $T_1$  is maintained constant (Fig. 3–30). The pipe is now insulated with a material whose thermal conductivity is  $k$  and outer radius is  $r_2$ . Heat is lost from the pipe to the surrounding medium at temperature  $T_\infty$ , with a convection heat transfer coefficient  $h$ . The rate of heat transfer from the insulated pipe to the surrounding air can be expressed as (Fig. 3–31)

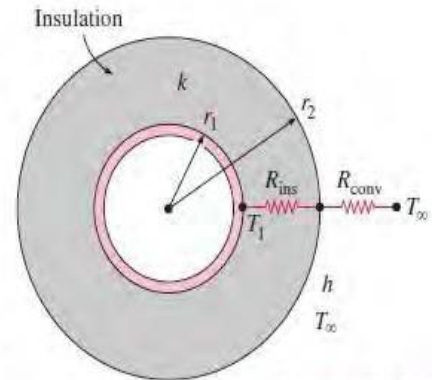
$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{ins}} + R_{\text{conv}}} = \frac{T_1 - T_\infty}{\frac{\ln(r_2/r_1)}{2\pi Lk} + \frac{1}{h(2\pi r_2 L)}} \quad (3-49)$$

The variation of  $\dot{Q}$  with the outer radius of the insulation  $r_2$  is plotted in Fig. 3–31. The value of  $r_2$  at which  $\dot{Q}$  reaches a maximum is determined from the requirement that  $d\dot{Q}/dr_2 = 0$  (zero slope). Performing the differentiation and solving for  $r_2$  yields the **critical radius of insulation** for a cylindrical body to be

$$r_{\text{cr, cylinder}} = \frac{k}{h} \quad (\text{m}) \quad (3-50)$$

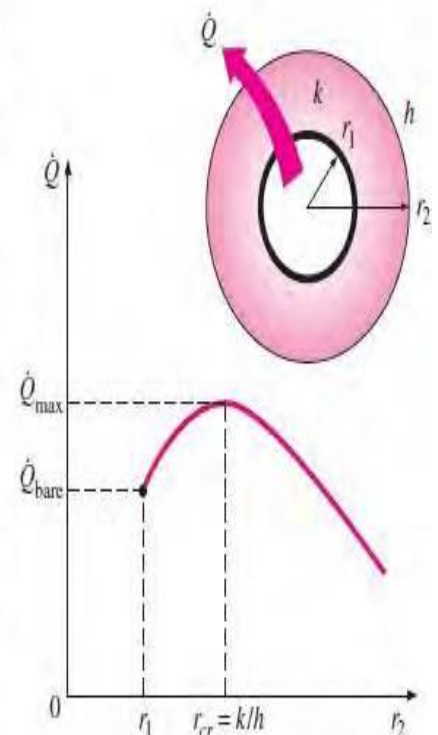
Note that the critical radius of insulation depends on the thermal conductivity of the insulation  $k$  and the external convection heat transfer coefficient  $h$ . The rate of heat transfer from the cylinder increases with the addition of insulation for  $r_2 < r_{\text{cr}}$ , reaches a maximum when  $r_2 = r_{\text{cr}}$ , and starts to decrease for  $r_2 > r_{\text{cr}}$ . Thus, insulating the pipe may actually increase the rate of heat transfer from the pipe instead of decreasing it when  $r_2 < r_{\text{cr}}$ .

The important question to answer at this point is whether we need to be concerned about the critical radius of insulation when insulating hot water pipes or even hot water tanks. Should we always check and make sure that the outer



**FIGURE 3–30**

An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.



**FIGURE 3–31**

radius of insulation exceeds the critical radius before we install any insulation? Probably not, as explained here.

The value of the critical radius  $r_{cr}$  will be the largest when  $k$  is large and  $h$  is small. Noting that the lowest value of  $h$  encountered in practice is about  $5 \text{ W/m}^2 \cdot ^\circ\text{C}$  for the case of natural convection of gases, and that the thermal conductivity of common insulating materials is about  $0.05 \text{ W/m} \cdot ^\circ\text{C}$ , the largest value of the critical radius we are likely to encounter is

$$r_{cr, \max} = \frac{k_{\max, \text{insulation}}}{h_{\min}} \approx \frac{0.05 \text{ W/m} \cdot ^\circ\text{C}}{5 \text{ W/m}^2 \cdot ^\circ\text{C}} = 0.01 \text{ m} = 1 \text{ cm}$$

This value would be even smaller when the radiation effects are considered. The critical radius would be much less in forced convection, often less than 1 mm, because of much larger  $h$  values associated with forced convection. Therefore, we can insulate hot water or steam pipes freely without worrying about the possibility of increasing the heat transfer by insulating the pipes.

The radius of electric wires may be smaller than the critical radius. Therefore, the plastic electrical insulation may actually *enhance* the heat transfer from electric wires and thus keep their steady operating temperatures at lower and thus safer levels.

The discussions above can be repeated for a sphere, and it can be shown in a similar manner that the critical radius of insulation for a spherical shell is

$$r_{cr, \text{sphere}} = \frac{2k}{h} \quad (3-51)$$

### Variable Thermal Conductivity:

You will recall from Chapter 1 that the thermal conductivity of a material, in general, varies with temperature (Fig. 2–62). However, this variation is mild for many materials in the range of practical interest and can be disregarded. In such cases, we can use an average value for the thermal conductivity and treat it as a constant, as we have been doing so far. This is also common practice for other temperature-dependent properties such as the density and specific heat.

When the variation of thermal conductivity with temperature in a specified temperature interval is large, however, it may be necessary to account for this variation to minimize the error. Accounting for the variation of the thermal conductivity with temperature, in general, complicates the analysis. But in the case of simple one-dimensional cases, we can obtain heat transfer relations in a straightforward manner.

When the variation of thermal conductivity with temperature  $k(T)$  is known, the average value of the thermal conductivity in the temperature range between  $T_1$  and  $T_2$  can be determined from

$$k_{\text{ave}} = \frac{\int_{T_1}^{T_2} k(T) dT}{T_2 - T_1} \quad (2-75)$$

This relation is based on the requirement that the rate of heat transfer through a medium with constant average thermal conductivity  $k_{\text{ave}}$  equals the rate of heat transfer through the same medium with variable conductivity  $k(T)$ . Note that in the case of constant thermal conductivity  $k(T) = k$ , Eq. 2–75 reduces to  $k_{\text{ave}} = k$ , as expected.



Then the rate of steady heat transfer through a plane wall, cylindrical layer, or spherical layer for the case of variable thermal conductivity can be determined by replacing the constant thermal conductivity  $k$  in Eqs. 2–57, 2–59, and 2–61 by the  $k_{\text{ave}}$  expression (or value) from Eq. 2–75:

$$\dot{Q}_{\text{plane wall}} = k_{\text{ave}} A \frac{T_1 - T_2}{L} = \frac{A}{L} \int_{T_2}^{T_1} k(T) dT \quad (2-76)$$

$$\dot{Q}_{\text{cylinder}} = 2\pi k_{\text{ave}} L \frac{T_1 - T_2}{\ln(r_2/r_1)} = \frac{2\pi L}{\ln(r_2/r_1)} \int_{T_2}^{T_1} k(T) dT \quad (2-77)$$

$$\dot{Q}_{\text{sphere}} = 4\pi k_{\text{ave}} r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1} = \frac{4\pi r_1 r_2}{r_2 - r_1} \int_{T_2}^{T_1} k(T) dT \quad (2-78)$$

The variation in thermal conductivity of a material with temperature in the temperature range of interest can often be approximated as a linear function and expressed as

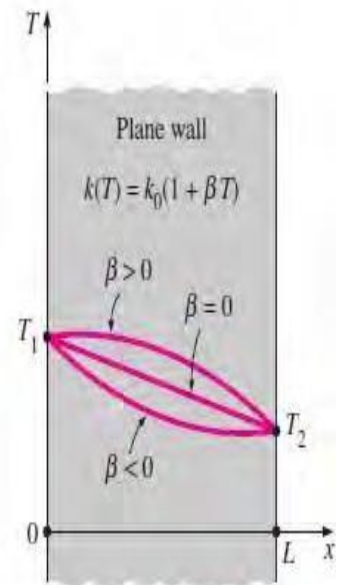
$$k(T) = k_0(1 + \beta T) \quad (2-79)$$

where  $\beta$  is called the **temperature coefficient of thermal conductivity**. The *average* value of thermal conductivity in the temperature range  $T_1$  to  $T_2$  in this case can be determined from

$$k_{\text{ave}} = \frac{\int_{T_1}^{T_2} k_0(1 + \beta T) dT}{T_2 - T_1} = k_0 \left( 1 + \beta \frac{T_2 + T_1}{2} \right) = k(T_{\text{ave}}) \quad (2-80)$$

Note that the *average thermal conductivity* in this case is equal to the thermal conductivity value at the *average temperature*.

We have mentioned earlier that in a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant. But this is no longer the case when the thermal conductivity changes with temperature, even linearly, as shown in Figure 2–63.



**FIGURE 2–63**

The variation of temperature in a plane wall during steady one-dimensional heat conduction for the cases of constant and variable thermal conductivity.

## UNIT-II

### Heat Transfer in Extended Surface (Fins)

#### Introduction

Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law:  $q = hA(T_s - T_\infty)$  temperature. Therefore, to increase the convective heat transfer, one can

- Increase the temperature difference ( $T_s - T_\infty$ ).
- Increase the convection coefficient  $h$ . This can be accomplished by increasing the fluid flow over the surface since  $h$  is a function of the flow velocity and the higher the velocity, the higher the  $h$ . Example: a cooling fan.
- Increase the contact surface area  $A$ . Example: a heat sink with fins.

Many times, when the first option is not in our control and the second option (i.e. increasing  $h$ ) is already stretched to its limit, we are left with the only alternative of increasing the effective surface area by using fins or extended surfaces. Fins are protrusions from the base surface into the cooling fluid, so that the extra surface of the protrusions is also in contact with the fluid. Most of you have encountered cooling fins on air-cooled engines (motorcycles, portable generators, etc.), electronic equipment (CPUs), automobile radiators, air conditioning equipment (condensers) and elsewhere.

#### Extended surface analysis

In this module, consideration will be limited to steady state analysis of rectangular or pin fins of constant cross sectional area. Annular fins or fins involving a tapered cross section may be analyzed by similar methods, but will involve solution of more complicated equations which result. Numerical methods of integration or computer programs can be used to advantage in such cases.

We start with the General Conduction Equation:

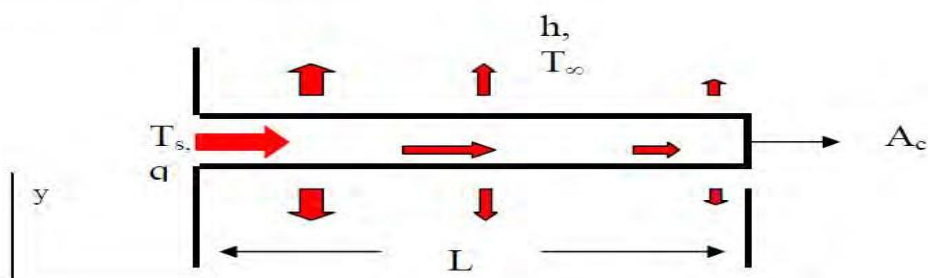
$$\frac{1}{\alpha} \cdot \frac{dT}{d\tau} \bigg|_{system} = \nabla^2 T + \frac{\dot{q}}{k} \quad (1)$$

After making the assumptions of Steady State, One-Dimensional Conduction, this equation reduces to the form:

$$\frac{d^2 T}{dx^2} + \frac{\dot{q}}{k} = 0 \quad (2)$$

This is a second order, ordinary differential equation and will require 2 boundary conditions to evaluate the two constants of integration that will arise.

Consider the cooling fin shown below:



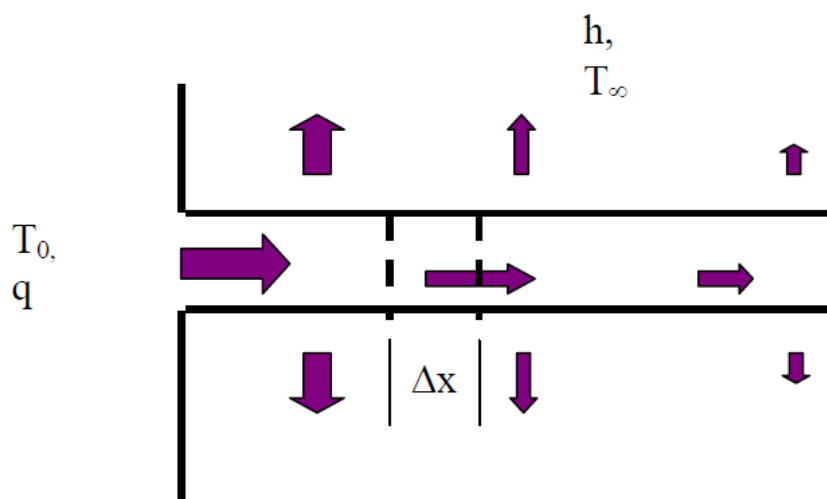


The fin is situated on the surface of a hot surface at  $T_s$  and surrounded by a coolant at temperature  $T_\infty$ , which cools with convective coefficient,  $h$ . The fin has a cross sectional area,  $A_c$ , (This is the area through which heat is conducted.) and an overall length,  $L$ .

Note that as energy is conducted down the length of the fin, some portion is lost, by convection, from the sides. Thus the heat flow varies along the length of the fin.

We further note that the arrows indicating the direction of heat flow point in both the  $x$  and  $y$  directions. This is an indication that this is truly a two- or three-dimensional heat flow, depending on the geometry of the fin. However, quite often, it is convenient to analyse a fin by examining an equivalent one-dimensional system. The equivalent system will involve the introduction of heat sinks (negative heat sources), which remove an amount of energy equivalent to what would be lost through the sides by convection.

Consider a differential length of the fin.



Across this segment the heat loss will be  $h \cdot (P \cdot \Delta x) \cdot (T - T_\infty)$ , where  $P$  is the perimeter around the fin. The equivalent heat sink would be  $-h \cdot (P \cdot \Delta x) \cdot (T - T_\infty)$ .

Equating the heat source to the convective loss:

$$\frac{-h \cdot P \cdot (T - T_{\infty})}{A_c} \quad (3)$$

Substitute this value into the General Conduction Equation as simplified for One-Dimension, Steady State Conduction with Sources:

$$\frac{d^2 T}{dx^2} - \frac{h \cdot P}{k \cdot A_c} \cdot (T - T_{\infty}) = 0 \quad (4)$$

which is the equation for a fin with a constant cross sectional area. This is the Second Order Differential Equation that we will solve for each fin analysis. Prior to solving, a couple of simplifications should be noted. First, we see that  $h$ ,  $P$ ,  $k$  and  $A_c$  are all independent of  $x$  in the defined system (They may not be constant if a more general analysis is desired.). We replace this ratio with a constant. Let

$$m^2 = \frac{h \cdot P}{k \cdot A_c} \quad (5)$$

then:

$$\frac{d^2 T}{dx^2} - m^2 \cdot (T - T_{\infty}) = 0 \quad (6)$$

Next we notice that the equation is non-homogeneous (due to the  $T_{\infty}$  term). Recall that non-homogeneous differential equations require both a general and a particular solution. We can make this equation homogeneous by introducing the temperature relative to the surroundings:

$$\theta \equiv T - T_{\infty} \quad (7)$$

Differentiating this equation we find:

$$\frac{d\theta}{dx} = \frac{dT}{dx} \quad (8)$$

Differentiate a second time:

$$\frac{d^2 \theta}{dx^2} = \frac{d^2 T}{dx^2} \quad (9)$$

Substitute into the Fin Equation:

$$\frac{d^2 \theta}{dx^2} - m^2 \cdot \theta = 0 \quad (10)$$

This equation is a Second Order, Homogeneous Differential Equation.

We apply a standard technique for solving a second order homogeneous linear differential equation.

Try  $\theta = e^{\alpha \cdot x}$ . Differentiate this expression twice:

$$\frac{d\theta}{dx} = \alpha \cdot e^{\alpha \cdot x} \quad (11)$$

$$\frac{d^2\theta}{dx^2} = \alpha^2 \cdot e^{\alpha \cdot x} \quad (12)$$

Substitute this trial solution into the differential equation:

$$\alpha^2 \cdot e^{\alpha \cdot x} - m^2 \cdot e^{\alpha \cdot x} = 0 \quad (13)$$

Equation (13) provides the following relation:

$$\alpha = \pm m \quad (14)$$

We now have two solutions to the equation. The general solution to the above differential equation will be a linear combination of each of the independent solutions.

Then:

$$\theta = A \cdot e^{m \cdot x} + B \cdot e^{-m \cdot x} \quad (15)$$

where A and B are arbitrary constants which need to be determined from the boundary conditions. Note that it is a 2<sup>nd</sup> order differential equation, and hence we need two boundary conditions to determine the two constants of integration.

An alternative solution can be obtained as follows: Note that the hyperbolic sin, sinh, the hyperbolic cosine, cosh, are defined as:

$$\sinh(m \cdot x) = \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} \quad \cosh(m \cdot x) = \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} \quad (16)$$

We may write:

$$C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) = C \cdot \frac{e^{m \cdot x} + e^{-m \cdot x}}{2} + D \cdot \frac{e^{m \cdot x} - e^{-m \cdot x}}{2} = \frac{C+D}{2} \cdot e^{m \cdot x} + \frac{C-D}{2} \cdot e^{-m \cdot x} \quad (17)$$

We see that if (C+D)/2 replaces A and (C-D)/2 replaces B then the two solutions are equivalent.

$$\theta = C \cdot \cosh(m \cdot x) + D \cdot \sinh(m \cdot x) \quad (18)$$

Generally the exponential solution is used for very long fins, the hyperbolic solutions for other cases.

### Boundary Conditions:

Since the solution results in 2 constants of integration we require 2 boundary conditions. The first one is obvious, as one end of the fin will be attached to a hot surface and will come into thermal equilibrium with that surface. Hence, at the fin base,

$$\theta(0) = T_0 - T_\infty \equiv \theta_0 \quad (19)$$

The second boundary condition depends on the condition imposed at the other end of the fin. There are various possibilities, as described below.

#### *Very long fins:*

For very long fins, the end located a long distance from the heat source will approach the temperature of the surroundings. Hence,

$$\theta(\infty) = 0 \quad (20)$$

Substitute the second condition into the exponential solution of the fin equation:

$$\theta(\infty) = 0 = A \cdot e^{m \cdot \infty} + B \cdot e^{-m \cdot 0} \quad (21)$$

The first exponential term is infinite and the second is equal to zero. The only way that this equation can be valid is if  $A = 0$ . Now apply the second boundary condition.

$$\theta(0) = \theta_0 = B \cdot e^{-m \cdot 0} \Rightarrow B = \theta_0 \quad (22)$$

The general temperature profile for a very long fin is then:

$$\theta(x) = \theta_0 \cdot e^{-m \cdot x} \quad (23)$$

If we wish to find the heat flow through the fin, we may apply Fourier Law:

$$q = -k \cdot A_c \cdot \frac{dT}{dx} = -k \cdot A_c \cdot \frac{d\theta}{dx} \quad (24)$$

Differentiate the temperature profile:

$$\frac{d\theta}{dx} = -\theta_0 \cdot m \cdot e^{-m \cdot x} \quad (25)$$

So that:

$$q = k \cdot A_c \cdot \theta_0 \cdot \left[ \frac{h \cdot P}{k \cdot A_c} \right]^{1/2} \cdot e^{-m \cdot x} = \sqrt{h \cdot P \cdot k \cdot A_c} \cdot e^{-m \cdot x} \cdot \theta_0 = M \theta_0 e^{-m \cdot x} \quad (26)$$

where  $M = \sqrt{h P k A_c}$ .

Often we wish to know the total heat flow through the fin, i.e. the heat flow entering at the base ( $x=0$ ).



# Unit-III

## Heat transfer by convection

### Forced convection:

The rate equation for the convective heat transfer (regardless of particular nature) between a surface and an adjacent fluid is prescribed by *Newton's law of cooling* (Refer Fig. 1.9)

$$Q = hA (t_s - t_f) \quad \dots(1.6)$$

where,

$Q$  = Rate of conductive heat transfer,

$A$  = Area exposed to heat transfer,

$t_s$  = Surface temperature,

$t_f$  = Fluid temperature, and

$h$  = Co-efficient of convective heat transfer.

The units of  $h$  are,

$$h = \frac{Q}{A (t_s - t_f)} = \frac{W}{m^2 \text{ } ^\circ\text{C}} \quad \text{or} \quad W/m^2 \text{ } ^\circ\text{C}$$

or,  $W/m^2K$

The coefficient of convective heat transfer ' $h$ ' (also known as *film heat transfer coefficient*) may be defined as "*the amount of heat transmitted for a unit temperature difference between the fluid and unit area of surface in unit time.*"

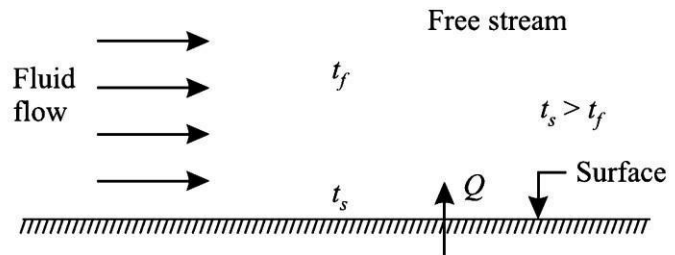
The value of ' $h$ ' depends on the following factors :

- (i) Thermodynamic and transport properties (e.g. viscosity, density, specific heat etc.).
- (ii) Nature of fluid flow.

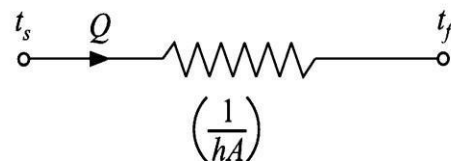
(iii) Geometry of the surface.

(iv) Prevailing thermal conditions.

Since ' $h$ ' depends upon several factors, it is difficult to frame a single equation to satisfy all the variations, however, by dimensional analysis an equation for the purpose can be obtained.



(a) Physical configuration



(b) Equivalent circuit

**Fig. 1.9.** Convective heat-transfer

The *mechanisms of convection* in which phase changes are involved lead to the important *fields of boiling and condensation*. Refer Fig. 1.9 (b). The quantity  $\frac{1}{hA} \left[ Q = \frac{t_s - t_f}{(1/hA)} \dots \text{Eqn (1.6)} \right]$  is called **convection thermal resistance**  $[(R_{th})_{conv}]$  to heat flow.

The following dimensionless numbers are used for the usual forced convection problems :

$$(i) \text{ Nusselt number, } Nu = \frac{hL}{k}; \quad (ii) \text{ Reynolds number, } Re = \frac{\rho LU}{\mu}$$

$$(iii) \text{ Prandtl number, } Pr = \frac{\mu c_p}{k}; \quad (iv) \text{ Stanton number, } St = \frac{h}{\rho c_p U}$$

In order to determine the value of convection coefficient  $h$ , the following conventional generalised basic equations are used :

$$Nu = f_1 (Re, Pr) = C_1 (Re)^m (Pr)^n; \quad St = f_2 (Re, Pr) = C_2 (Re)^a (Pr)^b$$

*The values (numerical) of the constants and exponents are determined through experiments. The properties of the fluid are evaluated on the basis of bulk temperature (unless stated otherwise).*

## LAMINAR FLOW OVER FLAT PLATES AND WALLS

The *local value of heat transfer coefficient* is given by

$$Nu_x = \frac{h_x x}{k} = 0.332 (Re_x)^{0.5} (Pr)^{0.333} \quad \dots \text{Blasius equation} \quad \dots$$

The *average value of heat transfer coefficient* is given by

$$Nu = \frac{\bar{h}L}{k} = 0.664 (Re_L)^{0.5} (Pr)^{0.333} \quad \dots$$

where,  $Re_x = \frac{Ux}{\nu}$ ,  $Re_L = \frac{UL}{\nu}$  and  $Pr = \frac{\mu c_p}{k}$

The above equations are valid for the following :

- (i) All fluids ( $Pr \geq 0.6$ ) except liquid metals
- (ii) Reynolds number  $Re \geq 40000$
- (iii) The fluid properties are evaluated at the mean film temperature,  $t_f = \frac{t_s + t_\infty}{2}$ .

For *liquid metals*, the following correlation has been proposed

$$Nu_x = 0.565 (Pe_x) \quad \dots$$

where  $Pe_x = Re_x \cdot Pr$

The above equation is valid for the following :

- (i)  $Pr \leq 0.05$
- (ii) The fluid properties are evaluated at the film temperature.

The local and average Nusselt numbers, from Colburn analogy, are given by :

$$Nu_x = 0.029 (Re_x)^{0.8} (Pr)^{1/3}$$

$$\overline{Nu} = 0.036 (Re_L)^{0.8} (Pr)^{1/3}$$

where the properties are evaluated at the mean film temperature.

When the flow lies in the *transition range*,

$$\overline{Nu} = 0.036 [(Re_L)^{0.8} - A] (Pr)^{1/3}$$

where

$$A = 18700 \text{ when } Re_c = 4 \times 10^5$$

$$A = 23100 \text{ when } Re_c = 5 \times 10^5$$

# UNIT-IV

## Boiling and condensation

**Boiling** is the convective heat transfer process that involves a phase change from liquid to vapour state. Boiling is also defined as evaporation at a solid-liquid surface. This is possible only when the temperature of the surface ( $t_s$ ) exceeds the saturation temperature corresponding to the liquid pressure ( $t_{sat}$ ). Heat is transferred from the solid surface to the liquid according to the law

$$Q = h A_s (t_s - t_{sat}) = h A_s \Delta t_e \quad \dots(9.1)$$

where,

$\Delta t_e = (t_s - t_{sat})$  is known as *excess temperature*.

The boiling process finds *applications* in the following cases:

- (i) Steam production (for generation of power and for industrial processes and space heating) in steam and nuclear power plants;
- (ii) Heat absorption in refrigeration and air conditioning systems;
- (iii) Distillation and refining of liquids;
- (iv) Concentration, dehydration and drying of foods and materials,
- (v) Cooling the machines like nuclear reactors and rocket motors where the large quantities of heat are released in relatively small volume (dissipation rates are as high as  $10^8 \text{ W/m}^2$ ; the maximum heat transfer rate in modern boiler is about  $2 \times 10^5 \text{ W/m}^2$ ).

The *boiling heat transfer phenomenon* may occur in the following forms:

### 1. *Pool boiling* :

In this case the liquid above the hot surface is essentially stagnant and its motion near the surface is due to free convection and mixing induced by bubble growth and detachment.

The pool boiling occurs in steam boilers involving *natural convection*.

### 2. *Forced convection boiling* :

This refers to a situation where the fluid motion is induced by external means (and also by free convection and bubble induced mixing). The liquid is pumped and forced to flow. This type of boiling occurs in water tube boilers involving forced convection.

### 3. *Sub-cooled or local boiling* :

In this case the liquid temperature is below the saturation temperature and



bubbles are formed in the vicinity of heat surface. These bubbles after travelling a short path get condensed in the liquid which has a temperature less than the boiling point.

#### 4. *Saturated boiling :*

Here, the liquid temperature exceeds the saturation temperature. The vapour bubbles formed at the solid surface (liquid-solid interface) are then propelled through the liquid by buoyancy effects and eventually escape from a free surface (liquid-vapour interface).

### **Boiling Regimes:**

The *process of boiling depends upon the nature of the surface, thermo-physical properties of the fluid and vapour bubble dynamics*. Due to involvement of large number of variables, general equations describing the boiling process are not available. Nonetheless, considerable progress has been made in arriving at a physical understanding of the boiling mechanism.

Figure 9.1 shows the temperature distribution in saturated pool boiling with a liquid-vapour interface. It is evident from the figure that although there is a sharp decline in the liquid temperature close to the solid surface, the temperature through most of the liquid remains slightly above saturation. Consequently bubbles generated at liquid-solid interface rise to and are transported across the liquid-vapour interface. Whether the boiling phenomenon corresponds to pool boiling or forced circulation boiling, there are three definite regimes of boiling (Interface evaporation, nucleate boiling and film boiling ) associated with progressively increasing heat flux, as shown in Fig. 9.2. This specific curve has been obtained from an electrically heated platinum wire submerged in a pool of water (at saturation temperature) by varying its surface temperature and measuring the surface heat flux  $q_s$ .

#### **1. Interface evaporation :**

Interface evaporation (evaporation process with no bubble formation) exists in region *I*, called the *free convection zone*. Here the excess temperature,  $\Delta t_e$ , is very small and  $\approx 5^\circ\text{C}$ . In this region the *liquid near the surface is superheated slightly, the convection currents circulate the liquid and evaporation takes place at the liquid surface*.

#### **2. Nucleate boiling :**

This type of boiling exists in regions *II* and *III*. With the increase in  $\Delta t_e$  (excess temperature) the formation of bubbles on the surface of the wire at certain localised spots commences. The bubbles condense in the liquid without reaching the liquid surface. In fact, it is the region *II* where nucleate boiling starts. With further increase in  $\Delta t_e$  the bubbles are formed more rapidly and rise to the surface of the liquid resulting in rapid evaporation, as indicated in the region *III*. The *nucleate boiling* is thus characterised by *formation of bubbles at the nucleation sites and the resulting liquid agitation. The bubble agitation induces considerable fluid mixing and that promotes substantial increase in the heat flux and the boiling heat transfer coefficient* (The equipment used for boiling should be designed to operate in this region only).

### 3. Film boiling :

Film boiling comprises of regions IV, V and VI. The trend of increase of heat flux with increase in excess temperature observed upto region III is *reversed* in region IV (called *film boiling region*). This is due to the fact that the bubble formation is very rapid and the bubbles blanket the heating surface and prevent the incoming fresh liquid from taking their place. Eventually the bubbles coalesce and form a vapour film which covers the surface completely. Since the thermal conductivity of vapour film is *much less* than that of the liquid the heat flux *drops* with growth in  $\Delta t_e$ . Within the temperature range  $50^\circ\text{C} < \Delta t_e < 150^\circ\text{C}$ , conditions oscillate between nucleate and film boiling and the phase is referred to as *transition boiling, unstable film boiling or partial film boiling* (region IV). With further increase in  $\Delta t_e$  the vapour film is stabilised and the heating surface is completely covered by a vapour blanket and the *heat flux* is the lowest as shown in region V. The surface temperatures required to maintain a stable film are high and under these conditions a sizeable amount of heat is lost by the surface *due to radiation*, as indicated in the region VI. The phenomenon of stable film boiling can be observed when a drop of *water falls on a red hot stove*. The drop does not evaporate immediately but dances a few times on the stove; this is *due to the formation of a stable steam film at the interface between the hot surface and the liquid droplet*.

*Critical heat flux or burnout point:* The critical heat flux or burnout point (Point A in Fig. 9.2) is *the point of maximum heat flux on the boiling curve at which transition from nucleate to film boiling initiates*. This point is also called the *boiling crisis* because the boiling process beyond that point is unstable unless of course, point B is reached. The temperature at point B is extremely high and normally above the melting of the solid. So if the heating of the metallic surface is not limited to point A, it is possible that the metal may get damaged or it may even melt (For this reason, point A is often termed as *boiling crisis or burnout point*). Thus we may be interested to operate the equipment close to this value and not beyond it.

The nucleate boiling is affected by the following factors:

#### 1. *Material, shape and condition of the heating surface :*

The boiling heat transfer coefficient depends greatly on the material of the heating surface; under identical conditions of pressure and temperature difference, it is different for different metals (*viz.* copper has high value than steel, zinc and chromium).

The heat transfer rates are also influenced by the *condition of heating surface*. A *rough surface gives a better heat transmission* than when the surface is either smooth or has been coated (smoothness weakens the metal tendency to get wetted).

The shape of the heating surface also affects transmission of heat.

2. *Liquid properties :*

Through experiments it has been observed that the size of the bubble increases with the dynamic viscosity of the liquid. With increase in bubble size, frequency of bubble formation decreases which results in reduced heat transfer.

Further, high thermal conductivity of the liquid improves the rate of heat transfer.

3. *Pressure :*

The pressure influences the rate of bubble growth and in turn also affects the temperature difference ( $t_s - t_\infty$ ) causing heat flow. For a boiling liquid, the maximum allowable heat flux first increases with pressure until critical pressure is reached and thereafter it declines.

4. *Mechanical agitation :*

Experiments have shown that the heat transfer rate increases with the increase in degree of agitation.

In boiling heat transfer, a *driving force is the excess temperature*, which is given by:

$$\Delta t_e = t_s - t_{sat} \quad \dots(9.7)$$

For the boiling process the governing equation is

$$Q = hA \Delta t_e$$

where  $h$  is the *boiling film coefficient*.

Since no analytical solution is available for boiling heat transfer due to difficult fluid behaviour, empirical relations are used for engineering calculations. some of them are given in following subsections.

## **Boiling correlations:**

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Since no analytical solution is available for boiling heat transfer due to difficult fluid behaviour, empirical relations are used for engineering calculations. some of them are given in following subsections.

## Nucleate boiling:

(i) For nucleate pool boiling, Rohsenow has recommended the following correlation:

$$q_s = \mu_l \cdot h_{fg} \left[ \frac{g (\rho_l - \rho_v)}{\sigma} \right]^{0.5} \left[ \frac{c_{pl} \cdot \Delta t_e}{C_{sl} \cdot h_{fg} \cdot Pr_l^n} \right]^3$$

where,

$q_s$  = Surface heat flux, W/m<sup>2</sup>;

$\mu_l$  = Liquid viscosity, kg/ms;

$h_{fg}$  = Enthalpy of vaporisation, J/kg;

$\rho_l$  = Density of saturated liquid, kg/m<sup>3</sup>;

$\rho_v$  = Density of the saturated vapour, kg/m<sup>3</sup>;

$\sigma$  = Surface tension of the liquid-vapour interface, N/m;

$c_{pl}$  = Specific heat of saturated liquid, J/kg K;

$\Delta t_e = (t_s - t_{sat})$  = excess temperature;

$C_{sl}$  = Surface fluid constant (determined from experimental data);

## Condensation:

The **condensation process** is the reverse of boiling process. The condensation sets in, whenever a saturation vapour comes in contact with a surface whose temperature is *lower than the saturation temperature* corresponding to the vapour pressure. As the vapour condenses, *latent heat is liberated* and there is flow of heat to the surface. The liquid condensate may get somewhat sub-cooled by contact with the cooled surface and that may eventually cause more vapour to condense on the exposed surface or upon the previously formed condensate.

Depending upon the condition of cool surface, condensation may occur in two possible ways: *Film condensation and dropwise condensation.*

### 1. Film Condensation :

*If the condensate tends to wet the surface and thereby forms a liquid film, then the condensation process is known as 'film condensation'. Here, the heat from the vapour to the cooling medium is transferred through the film of the condensate formed on the surface. The liquid flows down the cooling surface under the action of gravity and the layer continuously grows in thickness because of newly condensing vapours. The continuous film offers thermal resistance and checks further transfer of heat between the vapour and the surface.*

Further, the heat transfer from the vapour to the cooling surface takes place through the film formed on the surface. *The heat is transferred from the vapour to the condensate formed on the surface by 'convection' and it is further transferred from the condensate film to the cooling surface by the 'conduction'.* This combined mode of heat transfer by conduction and convection reduces the rates of heat transfer considerably (compared with dropwise condensation). That is the reason that heat transfer rates of filmwise condensation are *lower* than dropwise condensation. Fig. 9.6 (i) shows the film condensation on a vertical plate.

## 2. Dropwise condensation :

In 'dropwise condensation' *the vapour condenses into small liquid droplets of various sizes which fall down the surface in random fashion.* The drops form in cracks and pits on the surface, grow in size, break away from the surface, knock off other droplets and eventually run off the surface, without forming a film under the influence of gravity. Fig. 9.6 (ii) shows the dropwise condensation on a vertical plate.

In this type of condensation, a large portion of the area of solid surface is directly exposed to vapour without an insulating film of condensate liquid, consequently *higher heat transfer rate* (to the order of  $750 \text{ kW/m}^2$ ) *are achieved.* Dropwise condensation has been observed to occur either on highly polished surfaces, or on surfaces contaminated with impurities like fatty acids and organic compounds. This type of condensation gives coefficient of heat transfer generally *5 to 10 times larger than with film condensation.* Although dropwise condensation would be preferred to filmwise condensation yet *it is extremely difficult to achieve or maintain.* This is because most surfaces become 'wetted' after being exposed to condensing vapours over a period of time. Dropwise condensation can be obtained under controlled conditions with the help of certain additives to the condensate and various surface coatings but its commercial viability has not yet been approved. For this reason the *condensing equipment in use is designed on the basis of filmwise condensation.*

## Influence of Non condensable gases:

The presence of non-condensable gas such as air in a condensing vapour produces a detrimental effect on the heat transfer coefficient. It has been observed that even with a few percent by volume of air in steam the condensation heat transfer coefficient is reduced by more than fifty percent. This is owing to the fact that when a vapour containing non-condensable gas condenses, the non-condensable gas is left at the surface. Any further condensation at the surface will occur only after incoming vapour has diffused through this non-condensable gas collected in the vicinity of the surface. *The non-condensable gas adjacent to the surface acts as a thermal resistance to the condensation process. The rate of condensation decreases greatly when the condensable vapour is contaminated with even very small amounts of noncondensable gases.*

As the presence of non-condensable gas in a condensing vapour is undesirable, the general practice in the design of a condenser should be *to vent the noncondensable gas to the maximum extent possible.*

# UNIT-V

## Radiation Heat Transfer

**Radiation, energy transfer across a system boundary due to a change of Temperature, by the mechanism of photon emission or electromagnetic wave emission.**

The rate of emission of radiation by a body depends upon the following factors:

- (i) The temperature of the surface,
- (ii) The nature of the surface, and
- (iii) The wavelength or frequency of radiation.

The *parameters* which deal with the surface emission properties are given below :

- (i) **Total emissive power (E).** The “*emissive power*” is defined as the *total amount of radiation emitted by a body per unit area and time*. It is expressed in  $\text{W/m}^2$ . The *emissive power of a black body*, according to Stefan-Boltzmann, is *proportional to absolute temperature to the fourth power*.

$$E_b = \sigma T^4 \text{ W/m}^2$$

$$E_b = \sigma A T^4 \text{ W}$$

where,  $\sigma$  = Stefan-Boltzmann constant =  $5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4$

- (ii) **Monochromatic (spectral) emissive power ( $E_\lambda$ ).** It is often necessary to determine the spectral distribution of the energy radiated by a surface. At any given temperature the amount of radiation emitted per unit wavelength varies at different wavelengths. For this purpose the *monochromatic emissive power  $E_\lambda$*  of the surface is used. It is defined as *the rate of energy radiated per unit area of the surface per unit wavelength*.

The total emissive power is given by,

**Black body:** For perfectly absorbing body,  $\alpha = 1$ ,  $\rho = 0$ ,  $\tau = 0$ . Such a body is called a ‘*black body*’ (i.e., a *black body is one which neither reflects nor transmits any part of the incident radiation but absorbs all of it*). In practice, a perfect black body ( $\alpha = 1$ ) does not exist. However its concept is very important.

**Opaque body:** When no incident radiation is transmitted through the body, it is called an ‘*opaque body*’.

For the opaque body  $\tau = 0$ , and eqn. (11.5) reduces to

$$\alpha + \rho = 1 \quad \dots(11.6)$$

Solids generally do not transmit unless the material is of very thin section. Metals absorb radiation within a fraction of a micrometre, and insulators within a fraction of a millimetre. Glasses and liquids are, therefore, generally considered as opaque.



**White body:** If all the incident radiation falling on the body are reflected, it is called a ‘white body’.

For a white body,  $\rho = 1$ ,  $\alpha = 0$  and  $\tau = 0$ .

Gases such as hydrogen, oxygen and nitrogen (and their mixtures such as air) have a transmissivity of practically *unity*.

### Stefan Boltzmann’s Law:

i.e.,

$$E_b = \sigma T^4$$

where,

$E_b$  = Emissive power of a black body, and

$\sigma$  = Stefan-Boltzmann constant

$$= 5.67 \times 10^{-8} \text{ W/m}^2 \text{ K}^4.$$

Equation (11.7) can be rewritten as:

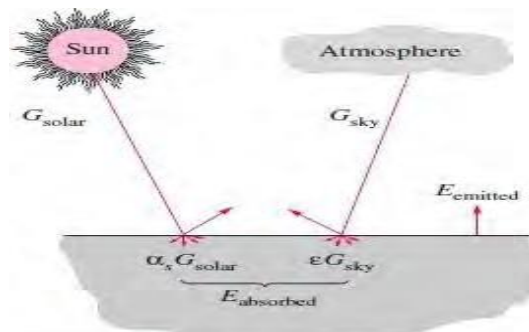
$$E_b = 5.67 \left( \frac{T}{100} \right)^4$$

### View Factor:

Radiation heat transfer between surfaces depends on the orientation of the surfaces relative to each other as well as their radiation properties and temperatures, as illustrated in Figure 12–1. For example, a camper will make the most use of a campfire on a cold night by standing as close to the fire as possible and by blocking as much of the radiation coming from the fire by turning her front to the fire instead of her side. Likewise, a person will maximize the amount of solar radiation incident on him and take a sunbath by lying down on his back instead of standing up on his feet. To account for the effects of orientation on radiation heat transfer between two surfaces, we define a new parameter called the view factor, which is a

purely geometric quantity and is independent of the surface properties and temperature. It is also called the shape factor, configuration factor, and angle factor. The view factor based on the assumption that the surfaces are diffuse emitters and diffuse reflectors is called the diffuse view factor, and the view factor based on the assumption that the surfaces are diffuse emitters but specular reflectors is called the specular view factor. In this book, we will consider radiation exchange between diffuse surfaces only, and thus the term view factor will simply mean diffuse view factor.

The view factor from a surface  $i$  to a surface  $j$  is denoted by  $F_{i \rightarrow j}$  or just  $F_{ij}$ , and is defined as  $F_{ij}$  – the fraction of the radiation leaving surface  $i$  that strikes surface  $j$  directly.



**FIGURE 11–43**

Radiation interactions of a surface exposed to solar and atmospheric radiation.



The relation for the spectral blackbody emissive power  $E_{b\lambda}$  was developed by Max Planck in 1901 in conjunction with his famous quantum theory. This relation is known as **Planck's law** and is expressed as

$$E_{b\lambda}(\lambda, T) = \frac{C_1}{\lambda^5 [\exp(C_2/\lambda T) - 1]} \quad (\text{W/m}^2 \cdot \mu\text{m}) \quad (11-4)$$

where

$$C_1 = 2\pi hc_0^2 = 3.742 \times 10^8 \text{ W} \cdot \mu\text{m}^4/\text{m}^2$$

$$C_2 = hc_0/k = 1.439 \times 10^4 \mu\text{m} \cdot \text{K}$$

Also,  $T$  is the absolute temperature of the surface,  $\lambda$  is the wavelength of the radiation emitted, and  $k = 1.38065 \times 10^{-23} \text{ J/K}$  is *Boltzmann's constant*. This relation is valid for a surface in a *vacuum* or a *gas*. For other mediums, it needs to be modified by replacing  $C_1$  by  $C_1/n^2$ , where  $n$  is the index of refraction of the medium. Note that the term *spectral* indicates dependence on wavelength.

## Absorptivity, Reflectivity, and Transmissivity

Everything around us constantly emits radiation, and the emissivity represents the emission characteristics of those bodies. This means that every body, including our own, is constantly bombarded by radiation coming from all directions over a range of wavelengths. Recall that radiation flux *incident on a surface* is called **irradiation** and is denoted by  $G$ .

When radiation strikes a surface, part of it is absorbed, part of it is reflected, and the remaining part, if any, is transmitted, as illustrated in Figure 11–31. *The fraction of irradiation absorbed by the surface* is called the **absorptivity**  $\alpha$ , *the fraction reflected by the surface* is called the **reflectivity**  $\rho$ , and *the fraction transmitted* is called the **transmissivity**  $\tau$ . That is,

$$\text{Absorptivity:} \quad \alpha = \frac{\text{Absorbed radiation}}{\text{Incident radiation}} = \frac{G_{\text{abs}}}{G}, \quad 0 \leq \alpha \leq 1 \quad (11-37)$$

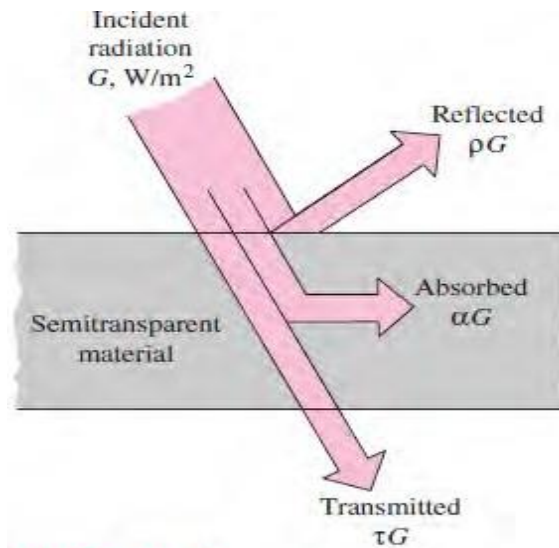
$$\text{Reflectivity:} \quad \rho = \frac{\text{Reflected radiation}}{\text{Incident radiation}} = \frac{G_{\text{ref}}}{G}, \quad 0 \leq \rho \leq 1 \quad (11-38)$$

$$\text{Transmissivity:} \quad \tau = \frac{\text{Transmitted radiation}}{\text{Incident radiation}} = \frac{G_{\text{tr}}}{G}, \quad 0 \leq \tau \leq 1 \quad (11-39)$$

where  $G$  is the radiation energy incident on the surface, and  $G_{\text{abs}}$ ,  $G_{\text{ref}}$ , and  $G_{\text{tr}}$  are the absorbed, reflected, and transmitted portions of it, respectively. The first law of thermodynamics requires that the sum of the absorbed, reflected, and transmitted radiation energy be equal to the incident radiation. That is,

$$G_{\text{abs}} + G_{\text{ref}} + G_{\text{tr}} = G \quad (11-40)$$

Dividing each term of this relation by  $G$  yields



**FIGURE 11-31**

The absorption, reflection, and transmission of incident radiation by a semitransparent material.

Dividing each term of this relation by  $G$  yields

$$\alpha + \rho + \tau = 1 \quad (11-41)$$

For opaque surfaces,  $\tau = 0$ , and thus

### Kirchhoff's Law

Consider a small body of surface area  $A_s$ , emissivity  $\varepsilon$ , and absorptivity  $\alpha$  at temperature  $T$  contained in a large isothermal enclosure at the same temperature, as shown in Figure 11-35. Recall that a large isothermal enclosure forms a blackbody cavity regardless of the radiative properties of the enclosure surface, and the body in the enclosure is too small to interfere with the blackbody nature of the cavity. Therefore, the radiation incident on any part of the surface of the small body is equal to the radiation emitted by a blackbody at temperature  $T$ . That is,  $G = E_b(T) = \sigma T^4$ , and the radiation absorbed by the small body per unit of its surface area is

$$G_{\text{abs}} = \alpha G = \alpha \sigma T^4$$

The radiation emitted by the small body is

$$E_{\text{emit}} = \varepsilon \sigma T^4$$

Considering that the small body is in thermal equilibrium with the enclosure, the net rate of heat transfer to the body must be zero. Therefore, the radiation emitted by the body must be equal to the radiation absorbed by it:

$$A_s \varepsilon \sigma T^4 = A_s \alpha \sigma T^4$$

Thus, we conclude that

$$\varepsilon(T) = \alpha(T) \quad (11-47)$$



That is, *the total hemispherical emissivity of a surface at temperature  $T$  is equal to its total hemispherical absorptivity for radiation coming from a black-body at the same temperature.* This relation, which greatly simplifies the radiation analysis, was first developed by Gustav Kirchhoff in 1860 and is now called **Kirchhoff's law**. Note that this relation is derived under the condition

that the surface temperature is equal to the temperature of the source of irradiation, and the reader is cautioned against using it when considerable difference (more than a few hundred degrees) exists between the surface temperature and the temperature of the source of irradiation.

The derivation above can also be repeated for radiation at a specified wavelength to obtain the *spectral* form of Kirchhoff's law:

$$\varepsilon_{\lambda}(T) = \alpha_{\lambda}(T) \quad (11-48)$$

This relation is valid when the irradiation or the emitted radiation is independent of direction. The form of Kirchhoff's law that involves no restrictions is the *spectral directional* form expressed as  $\varepsilon_{\lambda, \theta}(T) = \alpha_{\lambda, \theta}(T)$ . That is, the emissivity of a surface at a specified wavelength, direction, and temperature is always equal to its absorptivity at the same wavelength, direction, and temperature.

It is very tempting to use Kirchhoff's law in radiation analysis since the relation  $\varepsilon = \alpha$  together with  $\rho = 1 - \alpha$  enables us to determine all three properties of an opaque surface from a knowledge of only *one* property. Although Eq. 11-47 gives acceptable results in most cases, in practice, care should be exercised when there is considerable difference between the surface temperature and the temperature of the source of incident radiation.